Simulation of Resistivity and Sonic Borehole Logging Measurements Using hp Finite Elements

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overview

1. Motivation.


4. Electromagnetic Simulations.

5. Inverse Problems.

6. Conclusions.
OBJECTIVES: To determine payzones (porosity), amount of oil/gas (saturation), and ability to extract oil/gas (permeability).
motivation and objectives

Main Objective: To Solve a Multiphysics Inverse Problem

Software to solve the DIRECT problem is essential in order to solve the INVERSE problem.
method

The $h$-Finite Element Method

1. Convergence limited by the polynomial degree, and large material contrasts.
2. Optimal $h$-grids do NOT converge exponentially in real applications.
3. They may “lock” (100% error).

The $p$-Finite Element Method

1. Exponential convergence feasible for analytical (“nice”) solutions.
2. Optimal $p$-grids do NOT converge exponentially in real applications.
3. If initial $h$-grid is not adequate, the $p$-method will fail miserably.

The $hp$-Finite Element Method

1. Exponential convergence feasible for ALL solutions.
2. Optimal $hp$-grids DO converge exponentially in real applications.
3. If initial $hp$-grid is not adequate, results will still be great.
Energy norm based fully automatic $hp$-adaptive strategy

Coarse grids 
($hp$)

Fine grids 
($h/2, p + 1$)

SOL. METHOD ON FINE GRIDS:
A TWO GRID SOLVER

D. Pardo et. al. For more info, visit: www.bcamath.org/research/mip
Refinement strategy

Notation:

- $K$ is an element of the $h_p$-grid.
- $E_C = E_{hp}$ (coarse grid) $\prec E_{\hat{hp}} \prec E_F = E_{h/2,p+1}$ (fine grid).

The adaptive strategy maximizes the following quantity:

$$
\hat{h}_p = \arg\max_{\hat{h}_p} \sum_K \frac{|E_F - \Pi^K_{hp} E_F|_{?,K}^2 - |E_F - \Pi^K_{hp} E_F|_{?,K}^2}{(N_{\hat{hp}} - N_{hp})^2},
$$

where $\Pi^K_{hp} E_F$ is the projection based interpolation of solution $E_F$ over the $K$-th element of the $h_p$ grid.

The choice of the semi-norm depends upon the space in which the solution lives — $H^1$, $H(\text{curl})$, $H(\text{div})$ or $L^2$ —.
Projection based interpolation

\[ \Pi_{hp}^{K} E_F = E_{1}^{K,hp} + E_{2}^{K,hp} + E_{3}^{K,hp}. \]

- \( E_{1}^{K,hp} \) is the “bilinear vertex interpolant” of the \( K \)-th element of the \( hp \)-grid.
- \( E_{2}^{K,hp} \) is the “projection” of \( E_F - E_{1}^{K,hp} \) over each edge of the \( K \)-th element of the \( hp \)-grid.
- \( E_{3}^{K,hp} \) is the “projection” of \( E_F - E_{1}^{K,hp} - E_{2}^{K,hp} \) over the interior of the \( K \)-th element of the \( hp \)-grid.

The projection depends upon the space in which the solution lives —\( H^{1} \), \( H(\text{curl}) \), \( H(\text{div}) \) or \( L^{2} \)—.

Question: How can we combine energies coming from different norms/spaces?
De Rham diagram

De Rham diagram is critical to the theory of FE discretizations of multi-physics problems.

\[
\begin{array}{ccccccccc}
\mathbb{R} & \rightarrow & W & \nabla & Q & \nabla \times & V & \nabla \circ & L^2 & \rightarrow & 0 \\
\downarrow id & & \downarrow \Pi & & \downarrow \Pi^{\text{curl}} & & \downarrow \Pi^{\text{div}} & & \downarrow P \\
\mathbb{R} & \rightarrow & W^p & \nabla & Q^p & \nabla \times & V^p & \nabla \circ & W^{p-1} & \rightarrow & 0 .
\end{array}
\]

This diagram relates two exact sequences of spaces, on both continuous and discrete levels, and corresponding interpolation operators.
method

Mathematical Formulation (Goal-Oriented Adaptivity)

We consider the following problem (in variational form):

\[
\begin{aligned}
\text{Find } L(E), \text{ where } E \in V \text{ such that : } \\
b(E, \xi) = f(\xi) \quad \forall \xi \in V.
\end{aligned}
\]

We define residual \( r_e(\xi) = b(e, \xi) \). We seek for solution \( G \) of:

\[
\begin{aligned}
\text{Find } G \in V'' \sim V \text{ such that : } \\
G(r_e) = L(e).
\end{aligned}
\]

This is necessarily solved if we find the solution of the dual problem:

\[
\begin{aligned}
\text{Find } G \in V \text{ such that : } \\
b(E, G) = L(E) \quad \forall E \in V.
\end{aligned}
\]

Notice that \( L(e) = b(e, G) \).
Algorithm for Goal-Oriented Adaptivity

Solve DIRECT and DUAL problems on grid $hp$.  

$\rightarrow$  

Solve DIRECT and DUAL problems on grid $h/2, p + 1$.  

Compute $e = E_{h/2,p+1} - E_{hp}$, and $\epsilon = G_{h/2,p+1} - G_{hp}$.  

Represent the error as: $|L(e)| = |b(e, \epsilon)| \leq \sum_K |b_K(e, \epsilon)|$.  

Apply the fully automatic $hp$-adaptive algorithm.  

Solve DIRECT and DUAL problems on grid $hp$.  

$\rightarrow$  

Solve DIRECT and DUAL problems on grid $h/2, p + 1$.  

For more info, visit: www.bcamath.org/research/mip
**sonic simulations**

- **Acoustic domain:** borehole fluid \((c_f, \rho_f)\)
  \[
  \begin{aligned}
  i\omega p + c_f^2 \rho_f \nabla \cdot v &= 0 \\
  i\omega \rho_f v + \nabla p &= 0
  \end{aligned}
  \]
- **Elastic tool, casing, formation** \((V_p, V_s, \rho_s)\)
  \[
  0 = \nabla \cdot \sigma + \rho_s \omega^2 u
  \]
  \[
  \sigma = \lambda I \nabla \cdot u + \mu (\nabla u + \nabla^T u)
  \]
  \[
  \lambda = \rho_s (V_p^2 - 2V_s^2), \quad \mu = \rho_s V_s^2
  \]
- **Coupling:**
  \[
  n_f \cdot \nabla p = \rho_f \omega^2 n_f \cdot u
  \]
  \[
  n_s \cdot \sigma = -pn_s
  \]
- **Acoustic source:**
  \[
  n_f \cdot \nabla p = 1
  \]
sonic simulations

Verification of Sonic Measurements

- Formation thickness: 0.25 m.
- Monopole and dipole source, central frequency 8603 Hz.
sonic simulations

Monopole source — fast formation

Monopole source, fast formation, open borehole

Monopole source, fast formation with tool

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<th>P-wave</th>
<th>S-wave</th>
<th>Density</th>
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<tr>
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<td>$V_p [m/s]$</td>
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sonic simulations

Monopole source — fast formation — no tool

Normalized pressure, receivers offset, m

Time, ms
sonic simulations

Monopole source — fast formation — with tool

Normalized pressure, receivers offset, m

Time, ms
Sonic simulations

Monopole source — fast formation

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Where $V_p$ is the P-wave velocity, $V_s$ is the S-wave velocity, and $\rho$ is the density.
**Sonic Simulations**

**Dipole source — fast formation — no tool**

![Graph showing normalized pressure versus time for different receiver offsets.](image)

- **1D:** RX 1 — RX 8
- **2D:** RX 1 — RX 8

Normalized pressure, receivers offset, m

Time, ms

For more info, visit: [www.bcamath.org/research/mip](http://www.bcamath.org/research/mip)
sonic simulations

Dipole source — fast formation — with tool

1D: RX 1 — RX 2 — RX 3 — RX 4 — RX 5 — RX 6 — RX 7 — RX 8
2D: RX 1 — RX 2 — RX 3 — RX 4 — RX 5 — RX 6 — RX 7 — RX 8

Normalized pressure, receivers offset, m

Time, ms

Normalized pressure vs. time for different receivers (RX 1 to RX 8). The graph shows the response of the receivers to a dipole source in a 1D and 2D setting, with varying offsets. The peaks and waves indicate the arrival and interaction of the waves with the receivers.
sonic simulations

Dipole source — fast formation

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<td>1100</td>
<td>2200</td>
<td>7800</td>
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Dipole source, fast formation, no tool

Dipole source, fast formation with tool

Flexural mode:
- Analytical
- 1D semi-analytical
- 2D hp-FEM
sonic simulations

Direct calculation of dispersion curves

Dispersion curves contain the information about the slowness of the formation.

- dispersion curves are smooth with respect to variations in frequency,
- it is enough to calculate results only for a few frequencies (below 50)
- further reduction of the number of needed frequencies to 10 possible when only $V_p$ (high frequencies) and $V_s$ (low frequencies) are needed.

$\Rightarrow$ Dispersion curves are obtained directly from frequency domain results.

- no need to use in the simulations a Ricker wavelet (neither any other wavelet),
- the added stability of the problem,
- better performance of PML in frequency domain,
- smaller complexity of the problem.
sonic simulations

Direct calculation of dispersion curves

- Frequency spacing = 50Hz, N = 500
- Frequency spacing = 1250Hz, N = 20
- Frequency spacing = 500Hz, N = 50
- Frequency spacing = 2500Hz, N = 10
**Sonic Simulations**

Examples with layers

- **Formation thickness:** 0.25 m.
- **Monopole/dipole source:** Ricker wavelet, **central frequency** 8603 Hz.
sonic simulations

Layers (1) — monopole (waveforms)
sonic simulations

Layers (2) — monopole (waveforms)
electromagnetic simulations

3D Variational Formulation

Time-Harmonic Maxwell’s Equations

\[ \nabla \times H = \hat{\sigma}E + J^{imp} \quad \text{Ampere’s law} \ (\hat{\sigma} = \sigma + j\omega\epsilon) \]
\[ \nabla \times E = \hat{\mu}H + M^{imp} \quad \text{Faraday’s law} \ (\hat{\mu} = -j\omega\mu) \]
\[ \nabla \cdot (\epsilon E) = \rho \quad \text{Gauss’ law of Electricity} \]
\[ \nabla \cdot (\mu H) = 0 \quad \text{Gauss’ law of Magnetism} \]

E-VARIATIONAL FORMULATION:

Find \( E \in E_{\Gamma E} + H_{\Gamma E} (\text{curl}; \Omega) \) such that:

\[ \begin{aligned}
    \langle \nabla \times F, \hat{\mu}^{-1} \nabla \times E \rangle_{L^2(\Omega)} - \langle F, \hat{\sigma}E \rangle_{L^2(\Omega)} &= \langle F, J^{imp} \rangle_{L^2(\Omega)} \\
    - \langle F_t, J^{imp}_{\Gamma H} \rangle_{L^2(\Gamma_H)} + \langle \nabla \times F, \hat{\mu}^{-1} M^{imp} \rangle_{L^2(\Omega)} &= \forall F \in H_{\Gamma E} (\text{curl}; \Omega)
\end{aligned} \]
Dimensionality Reduction for Maxwell’s Equations
Solving a 3D problem is CPU time and memory intensive. In some cases, we may reduce the complexity of the problem by using Fourier analysis.

**Borehole Problems**
Cylindrical Coordinates
Fourier Series Expansion

\[ E(\phi) := \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \mathcal{F}_n(E) e^{jn\phi} \]

**X-Well, CSEM Problems**
Cartesian Coordinates
Fourier Transform

\[ E(x_1) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}_r(E) e^{jrx_1} dx_1 \]
**electromagnetic simulations**

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**Fourier Series Expansion**

**Fourier series expansion**

\[
\mathcal{F}_n(E) := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} E(\phi) e^{-jn\phi} d\phi ; \quad E(\phi) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \mathcal{F}_n(E) e^{jn\phi}.
\]

**Main properties**

- **Compatibility with differentiation** \( \mathcal{F}_n(\frac{\partial E}{\partial \phi}) = jn \mathcal{F}_n(E) \):

\[
\mathcal{F}_n(\nabla \times E) = \nabla^n \times (\mathcal{F}_n(E)),
\]

where

\[
\nabla^n \times E := \left( \frac{jn E_z}{\rho} - \frac{\partial E_\phi}{\partial z}, \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho}, \frac{1}{\rho} \frac{\partial (\rho E_\phi)}{\partial \rho} - \frac{jn E_\rho}{\rho} \right),
\]

- **\( L_2 \)-Orthogonality:**

\[
\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{jn\phi} e^{-jm\phi} d\phi = \sqrt{2\pi} \delta_{nm}.
\]
**electromagnetic simulations**

### Fourier Transform

**Fourier transform**

$$\mathcal{F}_r(E) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} E(x)e^{-jrx}dx$$  ;  
$$E(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}_r(E)e^{jrx}dr.$$  

**Inverse Fourier transform**

### Main properties

- **Compatibility with differentiation** \(\mathcal{F}_r(\frac{\partial E}{\partial x}) = jr\mathcal{F}_r(E):**

  $$\mathcal{F}_r(\nabla \times E) = \nabla^r \times (\mathcal{F}_r(E)),$$

  where

  $$\nabla^r \times E := \begin{pmatrix} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} - jrE_z, & jrE_y - \frac{\partial E_x}{\partial y} \end{pmatrix},$$

- **L$_2$-Orthogonality:**

  $$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{jrx}e^{-jsx} = \sqrt{2\pi}\delta_{sr}.$$
electromagnetic simulations

E-Variational Formulations (Cylindrical Coordinates)

FINITE ELEMENT —3D—:

\[
\begin{cases}
\text{Find } E \in E_{\Gamma_E} + H_{\Gamma_E}(\text{curl}; \Omega) \text{ such that:} \\
\langle \nabla \times F, \hat{\mu}^{-1} \nabla \times E \rangle_{L^2(\Omega)} - \langle F, \hat{\sigma} E \rangle_{L^2(\Omega)} = \langle F, J^{\text{imp}} \rangle_{L^2(\Omega)} \\
- \langle F_t, J^{\text{imp}}_{\Gamma_H} \rangle_{L^2(\Gamma_H)} + \langle \nabla \times F, \hat{\mu}^{-1} M^{\text{imp}} \rangle_{L^2(\Omega)} \quad \forall F \in H_{\Gamma_E}(\text{curl}; \Omega)
\end{cases}
\]

FOURIER FINITE ELEMENT —3D = Sequence of Coupled 2D Problems—:

\[
\begin{cases}
\text{Find } E = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \mathcal{F}_n(E) e^{jn\phi}, \text{ where for each } n: \\
\mathcal{F}_n(E) \in \mathcal{F}_n(E_{\Gamma_{E,1D}}) + H_{\Gamma_{E,1D}}(\text{curl}^n; \Omega_{2D}), \text{ and} \\
\sum_{m=-\infty}^{\infty} \langle \nabla^n \times \mathcal{F}_n(F), \mathcal{F}_{n-m}(\hat{\mu}^{-1}) \nabla^m \times \mathcal{F}_m(E) \rangle_{L^2(\Omega_{2D})} - \langle \mathcal{F}_n(F), \mathcal{F}_{n-m}(\hat{\sigma}) \mathcal{F}_m(E) \rangle_{L^2(\Omega_{2D})} \\
= \langle \mathcal{F}_n(F), \mathcal{F}_n(J^{\text{imp}}) \rangle_{L^2(\Omega_{2D})} - \langle \mathcal{F}_n(F_t), \mathcal{F}_n(J^{\text{imp}}_S) \rangle_{L^2(\Gamma_{H,1D})} \\
+ \sum_{m=-\infty}^{\infty} \langle \nabla^n \times \mathcal{F}_n(F), \mathcal{F}_{n-m}(\hat{\mu}^{-1}) \mathcal{F}_m(M^{\text{imp}}) \rangle_{L^2(\Omega_{2D})} \quad \forall \mathcal{F}_n(F) \in H_{\Gamma_{E,1D}}(\text{curl}^n; \Omega_{2D})
\end{cases}
\]
electromagnetic simulations

Cartesian system of coordinates: \( x = (x, y, z) \).

New non-orthogonal system of coordinates: \( \zeta = (\zeta_1, \zeta_2, \zeta_3) \).

\[
\begin{align*}
\text{Subdomain I} & : \quad \begin{cases} 
  x &= \zeta_1 \cos \zeta_2 \\
  y &= \zeta_1 \sin \zeta_2 \\
  z &= \zeta_3
\end{cases} \\
\text{Subdomain II} & : \quad \begin{cases} 
  x &= \zeta_1 \cos \zeta_2 \\
  y &= \zeta_1 \sin \zeta_2 \\
  z &= \zeta_3 + \tan \theta_0 \frac{\zeta_1 - \rho_1}{\rho_2 - \rho_1} \rho_2
\end{cases} \\
\text{Subdomain III} & : \quad \begin{cases} 
  x &= \zeta_1 \cos \zeta_2 \\
  y &= \zeta_1 \sin \zeta_2 \\
  z &= \zeta_3 + \tan \theta_0 \zeta_1
\end{cases}
\end{align*}
\]
electromagnetic simulations

E-Variational Formulation in the New System of Coordinates $\zeta$

In the new system of coordinates, we obtain:

3D FOURIER FINITE ELEMENT FORMULATION
— Sequence of “Weakly” Coupled 2D Problems —

\[
\begin{aligned}
\text{Find } E &= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \mathcal{F}_n(E) e^{jn\zeta}, \text{ where for each } n:\n
\mathcal{F}_n(E) &\in \mathcal{F}_n(E_{\Gamma E,1D}) + H_{\Gamma E,1D}(\text{curl}^n; \Omega_{2D}), \text{ and } \\
2 \sum_{m=-2}^{n} \left< \nabla^n \mathcal{F}_n(F), \mathcal{F}_{n-m}(\mu_{mod}^{-1}) \nabla^m \mathcal{F}_m(E) \right>_{L^2(\Omega_{2D})} - \left< \mathcal{F}_n(F), \mathcal{F}_{n-m}(\sigma_{mod}) \mathcal{F}_m(E) \right>_{L^2(\Omega_{2D})} \\
&= \left< \mathcal{F}_n(F), \mathcal{F}_n(J_{imp}) \right>_{L^2(\Omega_{2D})} - \left< \mathcal{F}_n(F_t), \mathcal{F}_n(J_{imp}^S) \right>_{L^2(\Gamma_{H,1D})} \\
&+ 2 \sum_{m=-2}^{n} \left< \nabla^n \mathcal{F}_n(F), \mathcal{F}_{n-m}(\mu_{mod}^{-1}) \mathcal{F}_m(M_{imp}) \right>_{L^2(\Omega_{2D})} \forall \mathcal{F}_n(F) \in H_{\Gamma E,1D}(\text{curl}^n; \Omega_{2D})
\end{aligned}
\]

Five Fourier modes are sufficient to represent EXACTLY the new material coefficients resulting from incorporating the change of coordinates.
electromagnetic simulations

E-Variational Formulations (Cylindrical Coordinates)

Assumption: For \( n \neq m \) we assume \( \mathcal{F}_{n-m}(\tilde{\mu}^{-1}) = \mathcal{F}_{n-m}(\tilde{\sigma}^{-1}) = 0 \).

**FOURIER FINITE ELEMENT —2.5D = Sequence of Uncoupled 2D Problems—:**

\[
\begin{aligned}
\text{Find } & E = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \mathcal{F}_n(E) \ e^{in\phi}, \text{ where for each } n: \\
& \mathcal{F}_n(E) \in \mathcal{F}_n(E_{\Gamma E,1D}) + H_{\Gamma E,1D}(\text{curl}^n; \Omega_{2D}), \text{ and} \\
& \left\langle \nabla^n \times \mathcal{F}_n(F) , \mathcal{F}_n(\tilde{\mu}^{-1}) \nabla^n \times \mathcal{F}_n(E) \right\rangle_{L^2(\Omega_{2D})} - \left\langle \mathcal{F}_n(F) , \mathcal{F}_n(\tilde{\sigma}) \mathcal{F}_n(E) \right\rangle_{L^2(\Omega_{2D})} \\
& = \left\langle \mathcal{F}_n(F) , \mathcal{F}_n(J^{\text{imp}}) \right\rangle_{L^2(\Omega_{2D})} - \left\langle \mathcal{F}_n(F_t) , \mathcal{F}_n(J^{\text{imp}}_S) \right\rangle_{L^2(\Gamma_{H,1D})} \\
& + \left\langle \nabla^n \times \mathcal{F}_n(F) , \mathcal{F}_n(\tilde{\mu}^{-1}) \mathcal{F}_n(M^{\text{imp}}) \right\rangle_{L^2(\Omega_{2D})} \quad \forall \mathcal{F}_n(F) \in H_{\Gamma E,1D}(\text{curl}^n; \Omega_{2D})
\end{aligned}
\]
electromagnetic simulations

2D Variational Formulation (Axi-symmetric Problems)
If we further assume that \( \mathcal{F}_n(J^{imp}) = \mathcal{F}_n(J_S^{imp}) = \mathcal{F}_n(M^{imp}) = 0 \quad \forall n \neq 0 \), then we obtain one uncoupled 2D problem. Now, \( E = \mathcal{F}_0(E) \).

\( E_\phi \) -Variational Formulation (Azimuthal)
\[
\begin{aligned}
\text{Find } E_\phi & \in E_{\phi,D} + \tilde{H}^1_D(\Omega) \text{ such that:} \\
\left\{ \begin{array}{l}
\langle \nabla \times F_\phi, \hat{\mu}_{\rho,z}^{-1} \nabla \times E_\phi \rangle_{L^2(\Omega_{2D})} - \langle F_\phi, \hat{\sigma}_\phi E_\phi \rangle_{L^2(\Omega_{2D})} = \langle F_\phi, J^{imp}_\phi \rangle_{L^2(\Omega_{2D})} \\
- \langle F_\phi, J^{imp}_\phi, \tilde{\Gamma}_H \rangle_{L^2(\tilde{\Gamma}_H)} + \langle F_\phi, \hat{\mu}_{\rho,z}^{-1} M^{imp}_{\rho,z} \rangle_{L^2(\Omega_{2D})} \quad \forall F_\phi \in \tilde{H}^1_D(\Omega)
\end{array} \right.
\end{aligned}
\]

\( E_{\rho,z} \) -Variational Formulation (Meridian)
\[
\begin{aligned}
\text{Find } E_{\rho,z} = (E_\rho, E_z) & \in E_D + \tilde{H}_D(\text{curl}; \Omega) \text{ such that:} \\
\left\{ \begin{array}{l}
\langle \nabla \times F_{\rho,z}, \hat{\mu}_\phi^{-1} \nabla \times E_{\rho,z} \rangle_{L^2(\Omega_{2D})} - \langle F_{\rho,z}, \hat{\sigma}_{\rho,z} E_{\rho,z} \rangle_{L^2(\Omega_{2D})} = \\
\langle F_{\rho,z}, J^{imp}_{\rho,z} \rangle_{L^2(\Omega_{2D})} - \langle (F_{\rho,z})_t, J^{imp}_{\rho,z}, \tilde{\Gamma}_H \rangle_{L^2(\tilde{\Gamma}_H)} \\
+ \langle F_{\rho,z}, \hat{\mu}_\phi^{-1} M^{imp}_\phi \rangle_{L^2(\Omega_{2D})} \quad \forall (F_\rho, F_z) \in \tilde{H}_D(\text{curl}; \Omega)
\end{array} \right.
\end{aligned}
\]
electromagnetic simulations

2D Finite Elements + 1D Fourier

3D Problem (using a Fourier Finite Element Method):
• $H(\text{curl})$ (Nedelec elements) for the meridian components ($E_{\rho,z}$), and
• $H^1$ (Lagrange elements) for the azimuthal component ($E_\phi$).

2.5D Problem (using a Fourier Finite Element Method):
• $H(\text{curl})$ (Nedelec elements) for the meridian components ($E_{\rho,z}$), and
• $H^1$ (Lagrange elements) for the azimuthal component ($E_\phi$).

2D Problem:
• $H(\text{curl})$ (Nedelec elements) in terms of the meridian components ($E_{\rho,z}$),
or
• $H^1$ (Lagrange elements) in terms of the azimuthal component ($E_\phi$).
electromagnetic simulations

Goal: To Study the Effect of Invasion, Anistotropy, and Magnetic Permeability.
electromagnetic simulations


Goal–Oriented hp–Adaptivity

Energy–norm hp–Adaptivity

Relative Error in %

Number of Unknowns N (scale $N^{1/3}$)

Upper bound for $|L(e)|/|L(u)|$

$|L(e)|/|L(u)|$

Energy–norm error

$|L(e)|/|L(u)|$
electromagnetic simulations

Goal-Oriented vs. Energy-norm $hp$-Adaptivity

Problem with Mandrel at 2 Mhz.

Continuous Elements (Goal-Oriented Adaptivity)

<table>
<thead>
<tr>
<th>Quantity of Interest</th>
<th>Real Part</th>
<th>Imag Part</th>
</tr>
</thead>
<tbody>
<tr>
<td>COARSE GRID</td>
<td>-0.1629862203E-01</td>
<td>-0.401694732E-02</td>
</tr>
<tr>
<td>FINE GRID</td>
<td>-0.1629862347E-01</td>
<td>-0.401694223E-02</td>
</tr>
</tbody>
</table>

Continuous Elements (Energy-Norm Adaptivity)

<table>
<thead>
<tr>
<th>Quantity of Interest</th>
<th>Real Part</th>
<th>Imag Part</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01% ENERGY ERROR</td>
<td>-0.1382759158E-01</td>
<td>-0.2989492851E-02</td>
</tr>
</tbody>
</table>

It is critical to use GOAL-ORIENTED adaptivity.
**Electromagnetic Simulations**

First. Vert. Diff. $E_\phi$ (solenoid). Position: 0.475m

**Energy Norm HP-Adaptivity**

<table>
<thead>
<tr>
<th>$p$</th>
<th>Value 1</th>
<th>Value 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.6851852</td>
<td>2.907408</td>
</tr>
<tr>
<td>2</td>
<td>-1.203704</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.018519</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
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<td>5</td>
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<td>8</td>
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</tr>
</tbody>
</table>

2Dhp90: A Fully automatic hp-adaptive Finite Element code
electromagnetic simulations

First. Vert. Diff. $E_\phi$ (solenoid). Position: 0.475m

GOAL-ORIENTED HP-ADAPTIVITY

2Dhp90: A Fully automatic hp-adaptive Finite Element code
electromagnetic simulations

First. Vert. Diff. $E_\phi$ (solenoid). Position: 0.475m

GOAL-ORIENTED HP-ADAPTIVITY (ZOOM)

2Dhp90: A Fully automatic hp-adaptive Finite Element code

1.7160494E-02  0.1369136

0.3603704

0.4344445
**electromagnetic simulations**

Simulation of Through Casing Resistivity Measurements

Left Figure:
- Axial-symmetric model
- One current electrode (emitter)
- Three voltage electrodes (collectors)

Objective:
- Compute second diff. of potential for various depth angles and possibly with water invasion

Method of solution:
- Fourier series expansion + change of coordinates + 2D goal-oriented hp-FEM
Through Casing Resistivity Measurements (Casing Conductivity)

Casing Resistivity $= 10^{-5} \Omega \cdot m$

Casing Resistivity $= 2.3 \times 10^{-7} \Omega \cdot m$

Qualitatively, results for various casing conductivities are similar even for deviated wells.
electromagnetic simulations

Through Casing Resistivity Measurements (Invasion)

30 degrees

60 degrees

Vertical position of voltage electrodes (m)

Second difference of potential (V)

10 cm INV

50 cm INV

NO INV

5 Ω·m

10000 Ω·m

1 Ω·m → 0.01 Ω·m

5 Ω·m

10 cm INV

50 cm INV

NO INV

5 Ω·m

10000 Ω·m

1 Ω·m → 0.01 Ω·m

5 Ω·m
electromagnetic simulations

Through Casing Resistivity Measurements (Invasion)

Second difference of potential (V)

Vertical position of voltage electrodes (m)

- 30 degrees
- 60 degrees

- NO INV
- 10 cm INV
- 50 cm INV

5 Ω·m
10 Ω·m
0.01 Ω·m
10kΩ·m

5 Ω·m
10 Ω·m
0.01 Ω·m
10kΩ·m

10^{-12} 10^{-10} 10^{-8} 10^{-6}

10^{-12} 10^{-10} 10^{-8} 10^{-6}
Model Problem and Verification

Finite Size Loop Antenna

Magnetic Buffer:
\[ \rho = 10000 \ \Omega \cdot m \]
\[ \mu_r = 10000 \]

Metallic Mandrel:
\[ \rho = 10^{-5} \Omega \cdot m \]
\[ \mu_r = 50 \]

Relative Error (in %)

LWD PROBLEM (30 degrees)
electromagnetic simulations

Dip Angle
LWD, 2 Mhz

Resistivity (Ω·m)

Real Part (V/m)

Imag Part (V/m)
**Electromagnetic Simulations**

Dip Angle + Invasion

LWD, 2 Mhz

- Resistivity ($\Omega \cdot m$)
- Real Part (V/m)
- Imag Part (V/m)

- 0 degrees
- 30 degrees
- 60 degrees
**electromagnetic simulations**

**Dip Angle + Anisotropy**

LWD, 2 Mhz

- Resistivity ($\Omega \text{m}$)
- Real Part (V/m)
- Imag Part (V/m)

- 0 degrees
- 30 degrees
- 60 degrees

VIRT.$\rho$

HORIZ.$\rho$
electromagnetic simulations

Dip Angle + Invasion + Anisotropy

LWD, 2 Mhz

Resistivity ($\Omega \cdot m$)

Real Part (V/m)

Imag Part (V/m)

NO INVASION

HORIZ. $\rho$

VERT. $\rho$

0 degrees

30 degrees

60 degrees

0 degrees

30 degrees

60 degrees
electromagnetic simulations

60-Degree Deviated Well

LWD, 2 Mhz

Resistivity (Ω·m)

Real Part (V/m)

Imag Part (V/m)

NO INV.
NO INV + ANI.
INV
INV + ANI
inversion problems

Variational Formulation (DC)

Notation:

\[ B(u, v; \sigma) = \langle \nabla v, \sigma \nabla u \rangle_{L^2(\Omega)} \quad \text{(bilinear \ } u, v) \]

\[ F_i(v) = \langle v, f_i \rangle_{L^2(\Omega)} + \langle v, g_i \rangle_{L^2(\partial \Omega)} \quad \text{(linear \ } v) \]

\[ L_i(u) = \langle l_i, u \rangle_{L^2(\Omega)} + \langle h_i, u \rangle_{L^2(\partial \Omega)} \quad \text{(linear \ } u) \]

Direct Problem (homogeneous Dirichlet BC’s):

\[
\begin{aligned}
\text{Find } \hat{u}_i \in V \text{ such that : } \\
B(\hat{u}_i, v; \sigma) = F_i(v) \quad \forall v \in V 
\end{aligned}
\]

Dual (Adjoint) Problem:

\[
\begin{aligned}
\text{Find } \hat{v}_i \in V \text{ such that : } \\
B(u, \hat{v}_i; \sigma) = L_i(u) \quad \forall u \in V 
\end{aligned}
\]
inversion problems

Variational Formulation (AC)

Notation:

\[ B(E, F; \sigma) = \langle \nabla \times F, \mu^{-1} \nabla \times E \rangle_{L^2(\Omega)} - \langle F, (\omega^2 \varepsilon - j\omega \sigma)E \rangle_{L^2(\Omega)} \]

\[ F_i(F) = -j\omega \langle F, J_i^{imp} \rangle_{L^2(\Omega)} + j\omega \langle F, J_{S,i}^{imp} \rangle_{L^2(\partial\Omega)} \]

\[ L_i(E) = \langle J_i^{adj}, E \rangle_{L^2(\Omega)} + \langle J_{S,i}^{adj}, E \rangle_{L^2(\partial\Omega)} \]

Direct Problem (homogeneous Dirichlet BC’s):

\[
\begin{align*}
\text{Find } \hat{E}_i \in W \text{ such that :} \\
B(\hat{E}_i, F; \sigma) = F_i(F) \quad \forall F \in W
\end{align*}
\]

Dual (Adjoint) Problem:

\[
\begin{align*}
\text{Find } \hat{F}_i \in W \text{ such that :} \\
B(E, \hat{F}_i; \sigma) = L_i(E) \quad \forall E \in W
\end{align*}
\]
inversion problems

Constrained Nonlinear Optimization Problem

Cost Functional:

\[
\begin{cases}
\text{Find } \sigma > 0 \text{ such that it minimizes } C_\beta(\sigma), \text{ where :} \\
C_\beta(\sigma) = ||W_m(L(\hat{u}_\sigma) - M)||^2_{l_2} + \beta||R(\sigma - \sigma_0)||^2_{L_2},
\end{cases}
\]

where

- \(M_i\) denotes the \(i\)-th measurement, \(M = (M_1, \ldots, M_n)\)
- \(L_i\) is the \(i\)-th quantity of interest, \(L = (L_1, \ldots, L_n)\)
- \(||M||^2_{l_2} = \sum_{i=1}^{n} M_i^2\) ; \(||R(\sigma - \sigma_0)||^2_{L_2} = \int (R(\sigma - \sigma_0))^2\)
- \(\beta\) is the relaxation parameter, \(\sigma_0\) is given, \(W_m\) are weights

Main objective (inversion problem): Find \(\hat{\sigma} = \min_{\sigma > 0} C_\beta(\sigma)\)
inversion problems

Solving a Constrained Nonlinear Optimization Problem

We select the following deterministic iterative method:

\[ \sigma^{(n+1)} = \sigma^{(n)} + \alpha^{(n)} \delta \sigma^{(n)} \]

- How to find a search direction \( \delta \sigma^{(n)} \)?
  - We will employ a change of coordinates and a truncated Taylor’s series expansion.

- How to determine the step size \( \alpha^{(n)} \)?
  - Either with a fixed size or using an approximation for computing \( L(\sigma^{(n)} + \alpha^{(n)} \delta \sigma^{(n)}) \).

- How to guarantee that the nonlinear constraints will be satisfied?
  - Imposing the Karush-Kuhn-Tucker (KKT) conditions or with a penalization method, or via a change of variables.
inversion problems

Search Direction Method

Change of coordinates:

\[ h(s) = \sigma \quad \Rightarrow \quad \text{Find } \hat{s} = \min_{h(s) > 0} C_{\beta}(s) \]

Taylor’s series expansion:

A) \[ C_{\beta}(s + \delta s) \approx C_{\beta}(s) + \delta s \nabla C_{\beta}(s) + 0.5\delta s^2 H_{C_{\beta}}(s) \]

B) \[ L(s + \delta s) \approx L(s) + \delta s \nabla L(s) , \quad R(s + \delta s) = R(s) + \delta s \nabla R(s) \]

Expansion A) leads to the Newton-Raphson method.

Expansion B) leads to the Gauss-Newton method.

Expansion A) with \( H_{C_{\beta}} = I \) leads to the steepest descent method.

Higher-order expansions require from higher-order derivatives.
inversion problems

Computation of Jacobian Matrix

Using the Fréchet Derivative:

\[
\frac{\partial L_i(\hat{u}_i)}{\partial s_j} = B \left( \frac{\partial \hat{u}_i}{\partial s_j}, \hat{v}_i, h(s) \right) + B \left( \hat{u}_i, \frac{\partial \hat{v}_i}{\partial s_j}, h(s) \right) + B \left( \hat{u}_i, \hat{v}_i, \frac{\partial h(s)}{\partial s_j} \right)
\]

Therefore, we conclude:

\[
\text{Jacobian Matrix} = \frac{\partial L_i(\hat{u}_i)}{\partial s_j} = -B \left( \hat{u}_i, \hat{v}_i, \frac{\partial h(s)}{\partial s_j} \right)
\]
inversion problems

Jacobian Function: One TX, one RX

2Dhp90: A Fully automatic hp-adaptive Finite Element code
inversion problems

Jacobian Function: One TX, one RX

2D hp90: A Fully automatic hp-adaptive Finite Element code
inversion problems

Jacobian Function: One TX, one RX

2Dhp90: A Fully automatic hp-adaptive Finite Element code
inversion problems

Computation of Hessian Matrix

Following a similar argument as for the Jacobian matrix, we obtain:

\[
\frac{\partial^2 L_i(\hat{u}_i)}{\partial s_j \partial s_k} = -B \left( \frac{\partial \hat{u}_i}{\partial s_j}, \hat{v}_i, \frac{\partial h(s)}{\partial s_k} \right) - B \left( \hat{u}_i, \frac{\partial \hat{v}_i}{\partial s_j}, \frac{\partial h(s)}{\partial s_k} \right) - B \left( \hat{u}_i, \hat{v}_i, \frac{\partial^2 h(s)}{\partial s_j \partial s_k} \right)
\]

How do we compute \( \frac{\partial \hat{u}_i}{\partial s_j} \) and \( \frac{\partial \hat{v}_i}{\partial s_j} \)?

Find \( \frac{\partial \hat{u}_i}{\partial s_j} \) such that:

\[
B \left( \frac{\partial \hat{u}_i}{\partial s_j}, v_i, h(s) \right) = -B \left( \hat{u}_i, v_i, \frac{\partial h(s)}{\partial s_j} \right) \quad \forall v_i
\]

Find \( \frac{\partial \hat{v}_i}{\partial s_j} \) such that:

\[
B \left( \frac{\partial \hat{v}_i}{\partial s_j}, u_i, h(s) \right) = -B \left( \hat{v}_i, u_i, \frac{\partial h(s)}{\partial s_j} \right) \quad \forall u_i
\]

We can compute the Hessian matrix EXACTLY by just solving our original problem for different right-hand-sides, and performing additional integrations.
inversion problems

Hessian Function: One TX, one RX

2Dhp90: A Fully automatic hp-adaptive Finite Element code
inversion problems

Hessian Function: One TX, two RXs

2Dhp90: A Fully automatic hp-adaptive Finite Element code
inversion problems

Hessian Function: One TX, three RXs

2D hp90: A Fully automatic hp-adaptive Finite Element code
**inversion problems**

Algorithms implemented within the inverse library

**JOINT MULTI-PHYSICS INVERSION LIBRARY**

- **CONSTRAINED OPTIMIZATION**
  - 1. KKT-Conditions
  - 2. Penalization Method
  - 3. Change of coordinates

- **SEARCH DIRECTIONS**
  - 1. Steepest Descent
  - 2. Gauss-Newton
  - 3. Newton-Raphson

- **STEP SIZE**
  - 1. Uniform step-size
  - 2. Variable step-size
    (multiple algorithms)

The inverse library is composed of multiple algorithms for imposing constraints, and finding search directions and corresponding step sizes.

Jacobian and Hessian matrices are computed exactly by simply solving the dual (adjoint) formulation and performing additional integrations.

The inverse library is compatible with multi-physics problems.
conclusions

- The $h_p$-Finite Element Method provides exponential convergence for a variety of multi-physics problems.

- We successfully employed a Fourier-Finite-Element method for simulation of electromagnetic and sonic logging measurements.

- We aim to perform joint-inversion of multiphysics measurements with a variety of applications (oil-industry, medicine, etc.).

- We need Ph.D. students, post-doctoral fellows and collaborators in order to solve this and other applications using advanced numerical methods.