On the detection of a moving obstacle in an ideal fluid by a boundary measurements

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This lecture is organized as follows:

- Existence of solutions for fluid-structure system
- Geometric inverse problem for viscous fluids
- Geometric inverse problem for inviscid fluids
Coupling between Euler’s equations and Newton’s laws

The equations modeling the dynamics of the fluid,

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = 0, \quad \text{in } \Omega(t) \times [0, T], \]
\[ \text{div } u = 0, \quad \text{in } \Omega(t) \times [0, T], \]
\[ u \cdot n = \left( h' + r(x - h)^\perp \right) \cdot n, \quad \text{on } \partial S(t) \times [0, T], \]

\( u \) and \( p \): velocity and pressure of the fluid
\( h(t) \): center of mass of \( S(t) \)
\( r \): angular velocity
\( n \): outward unit normal vector, \( x^\perp = (-x_2, x_1) \) if \( x = (x_1, x_2) \)
The rigid body is governed by the balance equations for linear and angular momentum (Newton’s laws)

\[ mh'' = \int_{\partial S(t)} p n d\Gamma, \quad \text{in } [0, T], \]  

(4)

\[ Jr' = \int_{\partial S(t)} (x - h(t))^\perp \cdot p n d\Gamma, \quad \text{in } [0, T], \]  

(5)

\[ u(x, 0) = a(x), \quad \forall x \in \Omega, \]  

(6)

\[ h(0) = 0 \in \mathbb{R}^2, \quad h'(0) = b \in \mathbb{R}^2, \quad r(0) = c \in \mathbb{R}. \]  

(7)
The liquid is assumed to be a perfect fluid; its dynamics is thus governed by incompressible Euler equations.

The continuity equation for the velocity (3) means that the fluid particles have the same velocity as the rigid body particles on $\partial S(t)$. In other words, the fluid does not enter into the rigid body.
The (positive) constants $m$ and $J$ are respectively the mass and the moment of inertia of the rigid body. They are defined by

$$m = \int_S \gamma \, dx, \quad J = \int_S \gamma |x|^2 \, dx,$$

where $\gamma$ denotes the (uniform) density of the rigid solid.

In Newton’s law (4) (resp., (5)), we notice that the only exterior force (resp., torque) applied to the rigid body is the one resulting from the fluid pressure integrated along the boundary $\partial S(t)$. 
We consider a reference geometry $S$ of class $C^1$ and piecewise $C^2$ (at time $t = 0$) for the rigid body:

and $\Omega(t) = \Omega \setminus S(t)$.

For the unbounded case $\Omega = \mathbb{R}^2$. 
Theorem

Let $\theta > 2$, $0 < \lambda < 1$, $a \in \mathcal{B}(\Omega) \cap H^1(\Omega)$, $b \in \mathbb{R}^2$, and $c \in \mathbb{R}$. Assume that $\text{div} a = 0$, $(a - b - cy^\perp) \cdot n|_{\partial S} = 0$, $\lim_{|y| \to +\infty} a(y) = 0$, and $\text{curl} a \in L^1_\theta(\Omega) \cap C^\lambda(\Omega)$.

Then there exists a solution $(v, q, l, r)$ of (1)-(6) such that:

- $v, \frac{\partial v}{\partial t}, \nabla v \in \mathcal{B}(\Omega_T)$, $\nabla q \in C(\Omega_T)$,
- $v \in C^1([0, T], L^2(\Omega)) \cap C([0, T], H^1(\Omega))$, $y^\perp \cdot \nabla v \in C([0, T], L^2(\Omega))$,
- $q \in C([0, T], \hat{H}^1(\Omega))$, $l \in C^1([0, T])$ and $r \in C^1([0, T])$,
- Such a solution is unique up to an arbitrary function of $t$ which may be added to $q$. 
Let $S$ the region occupied by the solid at $t = 0$ and by $\Omega = \mathbb{R}^2 \setminus \overline{S}$ the initial domain occupied by the fluid. Then, considering a suitable change of variables, we obtain an approximated Navier-Stokes system:

\[
\frac{\partial v}{\partial t} - \nu \Delta v + \left[ (v - l - ry^\perp) \cdot \nabla \right] v + rv^\perp + \nabla q = 0, \text{ in } Q_T \quad (8)
\]
\[
\text{div } v = 0, \text{ in } Q_T \quad (9)
\]
\[
v \cdot n = (l + ry^\perp) \cdot n, \text{ on } \Sigma_T \quad (10)
\]
\[
\text{curl } v = 0, \text{ on } \Sigma_T \quad (11)
\]
Moreover

\[ ml' = \int_{\partial S} qn \, d\Gamma - mrl^\perp, \quad \text{in } [0, T] \quad (12) \]
\[ Jr' = \int_{\partial S} qn \cdot y^\perp \, d\Gamma, \quad \text{in } [0, T] \quad (13) \]
\[ v(y, 0) = a(y), \quad \text{in } \Omega, \quad (14) \]
\[ l(0) = b \quad \text{and} \quad r(0) = c. \quad (15) \]
Vorticity estimates

- \( \omega^R_\nu := \text{curl} v^R_\nu \) be the vorticity.
- \( \omega^R_\nu \in L^2(0, T; H^1_0(\Omega)) \cap C([0, T]; L^2(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)) \).
- The vorticity verifies

\[
\frac{\partial \omega^R_\nu}{\partial t} - \nu \Delta \omega^R_\nu + (v^R_\nu - l^R_\nu - r^R_\nu y_\perp) \cdot \nabla \omega^R_\nu - r^R_\nu D_R(y) \cdot \nabla v^R_\nu = 0 \quad (16)
\]

- \( \omega^R_\nu = 0 \) on \( \partial S \times [0, T] \) and the initial condition \( \omega^R_\nu(0) = \text{curl} \ a \) in \( \Omega \).
• It is possible to obtain suitable estimates for the velocity $v$ and the vorcity $\omega$ which allow us to pass to the limit in $R \uparrow +\infty$ and $\nu \downarrow 0^+$.

• The unicity is a direct consequence of an energy estimate and Gronwall’s Lemma.
A. MUNNIER & J. G. HOUOT (2008) studied the bounded domain case in the case when the fluid is irrotational, that is, the velocity has the form \( v = \nabla \varphi \). The idea of the proof is to reduce the problem to study the existence of solutions for an ODE system.

C. Rosier & L. Rosier (2008) obtained a generalization of the potential case in the case of a ball, the proof is based in a different abstract approach.
Inverse geometrical problem

- Recover geometrical information (position and shape) about an a-priori unknown body $D$ immersed in an incompressible flow by means of exterior measurements.

**Figure 1.** Moving obstacle in a pipeline.
Viscous flow

• To this end, we perform measurements (on velocity $v$ and stress forces $\sigma(v, p)n$) along the boundary of the cavity $\Omega$ fulfilled by the liquid.

• Non-steady incompressible Stokes or Navier-Stokes system for the liquid.

• Non-homogeneous Dirichlet boundary condition on $\partial\Omega$ ($u = g$ on $\partial\Omega$).

• Non-slip condition on $\partial D$ ($u = $ on $\partial D$).
Let $\Omega$ be a smooth bounded set in $\mathbb{R}^N$ and let $D \subset \subset \Omega$ be an unknown rigid body immersed in the liquid.

Let $\phi \in H^{\frac{1}{2}}(\partial \Omega)$ be a non homogeneous Dirichlet boundary data and let $(v, p)$ be the solution of Stokes equations in $\Omega^* := \Omega \setminus \overline{D}$

$$
\begin{align*}
\text{(P)} & \quad \left\{ 
\begin{array}{ll}
\text{div}(\sigma(v, p)) &= 0 \quad \text{in } \Omega^* \\
\text{div } v &= 0 \quad \text{in } \Omega^* \\
v &= \phi \quad \text{on } \partial \Omega \\
v &= 0 \quad \text{on } \partial D
\end{array}
\right.
\end{align*}
$$
Let $\sigma(v, p)$ be the linear stress tensor, defined by

$$\sigma(v, p) = -pl + 2\nu e(v) \quad \text{where} \quad e(v) = \frac{(\nabla v + (\nabla v)^T)}{2}$$

The set of admissible bodies $D$ is

$$\mathcal{U}_{ad} = \{ D \subset \subset \Omega \quad : \quad D \text{ is a smooth, open set, such that, } \Omega \setminus \overline{D} \text{ is connected} \}$$
Let $\Lambda$ be the following boundary map, velocity to stress tensor

$$\Lambda : D \longrightarrow \Lambda_D$$

defined as follows

$$\Lambda_D(\phi) = \sigma(v, p)n \quad \text{on } \Gamma \subset \partial\Omega;$$

$(v, p)$ being the solution of the Stokes system $(P)$.

Our Inverse Problem is to recover $D$ from the above boundary map velocity to stress tensor

$$\Lambda : (D, \phi) \longrightarrow \Lambda_D(\phi)$$
Main results

- **Identifiability result**, that is, the injectivity of the velocity to stress tensor map $\Lambda$:

$$D_1 \neq D_2 \implies \sigma(v_1, p_1)n \neq \sigma(v_2, p_2)n \quad \text{on } \Gamma, \ \forall \phi.$$  

The proof is based in the unique continuation property for Stokes system (C. Fabre - G. Lebeau).
Stability result, that is, the continuity of the inverse of the velocity to stress tensor map $\Lambda$ (if two measures are close each other, then the rigid bodies are also close).

Algorithm and numerical results allow us to recover the volume and position of the unknown rigid body.
Numerical reconstruction

\[\text{div}(\sigma(\mathbf{v}_u, p_u)) = 0 \quad \text{in } \Omega^* + u\]
\[\text{div } \mathbf{v}_u = 0 \quad \text{in } \Omega^* + u\]
\[\mathbf{v}_u = \phi \quad \text{on } \Gamma_{in}\]
\[\mathbf{v}_u = 0 \quad \text{sur } \Gamma \cup \Gamma_m\]
\[\sigma(\mathbf{v}_u, p_u)n = 0 \quad \text{on } \Gamma_{out}\]
\[\mathbf{v}_u = 0 \quad \text{on } \partial D + u.\]
Let $\sigma_D = \Lambda_D(\phi)$ be the stress linear tensor on the external boundary $\Gamma_m$ measured for the unknown body $D$.

Minimization problem

$$
\min_{u \in \mathcal{U}_{ad}} \ J(u) = \min_{u \in \mathcal{U}_{ad}} \int_{\Gamma} \left| (\sigma(v_u, p_u)n - \Lambda_D(\phi)) \right|^2 ds.
$$

which has a unique global minimum.
The gradient of our objective function is:

\[ J'(u; w) = \int_{\partial D+u} (w \cdot n) \frac{\partial \mathbf{v}_u}{\partial \mathbf{n}} \cdot \sigma(\zeta, q)n ds, \]

\((\zeta, q)\) being the unique solution of the adjoint problem

\[
\begin{cases}
\text{div}(\sigma(\zeta, q)) = 0 & \text{in } \Omega^* + u \\
\text{div } \zeta = 0 & \text{in } \Omega^* + u \\
\sigma(\zeta, q)n = 0 & \text{on } \Gamma_{out} \\
\zeta = 2 [\sigma(\mathbf{v}_u, p_u)n - \Lambda_D(\phi)] & \text{on } \Gamma_m \\
\zeta = 0 & \text{on } \Gamma \cup \Gamma_{in} \\
\zeta = 0 & \text{on } \partial D + u.
\end{cases}
\]
Non convex and nonlinear objective function.

Steepest descent algorithm (SD) and non linear conjugate gradient (NLCG) method (an explicit formula for the gradient is available).
On the detection of a moving obstacle
Euler equations do not exhibit unique continuation property because of the existence of **ghost solutions**.

Let us consider \( \mathbf{v}(x) = (\partial \psi / \partial x_2, -\partial \psi / \partial x_1) \), where \( \psi \) is given by

\[
\psi(x) = -\int_1^{|x|} \frac{1}{r} \left( \int_1^r s \, w(s) \, ds \right) \, dr
\]

and the vorticity \( w \in C^\infty(\mathbb{R}^+) \) is chosen so that \( w(s) = 0 \) for \( s \geq 1 \) and \( \int_r^1 s \, w(s) \, ds = 0 \) for \( r \in (0, r_0) \), where \( r_0 \in (0, 1) \).
Let us fix $t = t_0$ and merely focus on the determination of the position and the velocity of $S(t_0)$ from a boundary measurement of the velocity of the fluid at time $t_0$.

We will restrict ourselves to potential flows, i.e. flows for which

$$\mathbf{v} = \nabla \varphi$$

(Here, $\varphi = \varphi(x, t)$ is the so-called velocity potential; it is a scalar unknown)

and to spherical obstacles

$$S(t) = B_1(h(t)) = \text{ball of radius 1 and centered at } h(t)$$
\[ \nabla \left( \frac{\partial \varphi}{\partial t} + \frac{1}{2} |\nabla \varphi|^2 + p \right) = 0 \quad \text{in} \quad \Omega \setminus S(t) \]

\[ \Delta \varphi = 0 \quad \text{in} \quad \Omega \setminus S(t) \]

\[ \frac{\partial \varphi}{\partial n} = (h' + \omega (x - h)^\perp) \cdot n \quad \text{on} \quad \partial S(t) \]

\[ \frac{\partial \varphi}{\partial n} = g \quad \text{on} \quad \partial \Omega \]

and for all \( t \geq 0 \).
Case \((h', \omega) = (0, 0)\) In the static case, the boundary condition on \(\partial S(t)\) simplifies to
\[
\frac{\partial \varphi}{\partial n} = 0,
\]
so the detection of \(S\) reduces to a quite classical problem:

What is new here?

Essentially, the fact that the obstacle is moving, i.e.,

$$(h', \omega) \neq (0,0)$$

- The velocity of $S$ being unknown, the classical argument based upon the unique continuation property for Laplace equation is not sufficient to derive the identifiability property.
- Indeed, the obstacle $S$ may occupy different positions and undergo different velocities for a given Neumann data $g$. 
Assume that $S(t) = B_1(h(t))$. Then

$$x - h = -n, \quad \forall x \in \partial B_1(h(t)),$$

and hence $(x - h) \perp \cdot n = 0$. Setting $l = h'$, the system reads

$$\begin{cases}
\Delta \varphi = 0 \quad \text{in} \quad \Omega \setminus \overline{B_1(h(t))} \\
\frac{\partial \varphi}{\partial n} = l \cdot n \quad \text{on} \quad \partial B_1(h(t)) \\
\frac{\partial \varphi}{\partial n} = g \quad \text{on} \quad \partial \Omega
\end{cases}$$

FIGURE 1. Moving obstacle in a pipeline.
Linear input flows are excluded

Clearly, \( g(x) = l \cdot n(x) \), with \( l \in \mathbb{R}^2 \) a given vector, has to be excluded, for it may lead to the situation where the ball, which is surrounded by a fluid flowing at the same velocity \( (\varphi(x, t) = l \cdot x) \), is not identifiable.
Let us consider the problems

\[ (P) \quad \begin{cases} 
\Delta \varphi_i = 0 \quad \text{in} \quad \Omega \setminus \overline{B_i} \\
\frac{\partial \varphi_i}{\partial n} = g \quad \text{on} \quad \partial \Omega \\
\frac{\partial \varphi_i}{\partial n} = l_i \cdot \mathbf{n} \quad \text{on} \quad \partial B_i 
\end{cases} \]
Theorem

Let $V$ be the following two-dimensional space

$$V = \text{Span} \{ e_1 \cdot n, e_2 \cdot n \} \subset L^\infty(\partial \Omega)$$

Assume that $g \in H^s(\partial \Omega) \setminus V$ with $s > 1/2$. For $i = 1, 2$, pick any $(h_i, l_i) \in \Omega_{ad} \times \mathbb{R}^2$ and let $\varphi_i$ denote the solution of $(P)$. Then problem $(P)$ is identifiable in the following sense:

$$\varphi_1 = \varphi_2 \text{ on } \Gamma_m \Rightarrow h_1 = h_2 \text{ and } l_1 = l_2.$$

Moreover, the following linear stability estimate holds:

$$\|\varphi_1 - \varphi_2\|_{H^{s+1}(\Gamma_m)/\mathbb{R}} \geq C \|(h_1 - h_2, l_1 - l_2)\|_{\mathbb{R}^4}.$$
Sketch of the proof

• By standard regularity results for elliptic problems \( \varphi_i \in H^{s+3/2}(\Omega \setminus \overline{B_i}) \subset C^1(\overline{\Omega} \setminus B_i) \). Assume that \( \varphi_1 = \varphi_2 \) on \( \Gamma_m \).

• Since \( \frac{\partial \varphi_1}{\partial n} = g = \frac{\partial \varphi_2}{\partial n} \) on \( \Gamma_m \), we infer (from the unique continuation property) that

\[
\varphi_1 = \varphi_2 \quad \text{in} \quad \Omega \setminus \overline{B_1 \cup B_2}.
\]

• We can thus define a function \( \varphi : \Omega \setminus \overline{B_1 \cap B_2} \to \mathbb{R} \) by

\[
\varphi(x) := \begin{cases} 
\varphi_1(x) & \text{if } x \in \Omega \setminus \overline{B_1} \\
\varphi_2(x) & \text{if } x \in \Omega \setminus \overline{B_2}
\end{cases}
\]
Then $\varphi$ verifies

$$\Delta \varphi = 0 \quad \text{in } \Omega \setminus \overline{B_1 \cap B_2}, \quad (17)$$

$$\frac{\partial \varphi}{\partial n} = g \quad \text{on } \partial \Omega, \quad (18)$$

$$\frac{\partial \varphi}{\partial n} = l_1 \cdot n \quad \text{on } \partial B_1, \quad (19)$$

$$\frac{\partial \varphi}{\partial n} = l_2 \cdot n \quad \text{on } \partial B_2. \quad (20)$$
If $B_1 \cap B_2 = \emptyset$, then

- $\varphi$ is as $\varphi_2$ defined and harmonic in $B_1$.

- From (19) we have that $\varphi = l_1 \cdot x + \text{const}$ in $B_1$.

- Therefore $g = l_1 \cdot n$ on $\partial \Omega$, which is a contradiction.
If $B_1 \cap B_2 \neq \emptyset$, and $l_1 = l_2 = l$,

- Let $D_1 = B_1 \setminus \overline{B_2}$, then $\varphi$ solves

$$
\Delta \varphi = 0 \quad \text{in} \quad D_1, \tag{21}
$$

$$
\frac{\partial \varphi}{\partial n} = l \cdot n \quad \text{on} \quad \partial D_1, \tag{22}
$$

- Thus $\varphi(x) = l \cdot x + \text{const}$

- Therefore $g \in V$, which is a contradiction.
If $B_1 \cap B_2 \neq \emptyset$, and $l_1 \neq l_2$, we have

- It is easy to see that $(l_2 - l_1) \cdot (h_2 - h_1) = 0$.
- Translating and rotating $\Omega$ if needed, we may assume that $h_1 = (0, \delta)$, $h_2 = (0, -\delta)$, with $0 < \delta < 1$, and $l_2 - l_1 = (\lambda, 0)$ for some $\lambda \neq 0$. 

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**Diagram:**

The diagram illustrates a scenario with a circle and points labeled $i(1+\delta)$, $-i(1+\delta)$, $M_-$, and $M_+$. The circle is shaded, and the points are located on the circle and along the real axis.

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On the detection of a moving obstacle
We can see that $\varphi$ satisfies the problem

\begin{align}
\Delta \varphi &= 0 \quad \text{in } D_1, \\
\frac{\partial \varphi}{\partial n} &= g \quad \text{on } \partial \Omega, \\
\frac{\partial \varphi}{\partial n} &= e_1 \cdot n \quad \text{on } \Gamma_1 := (\partial B_1) \setminus B_2, \\
\frac{\partial \varphi}{\partial n} &= -e_1 \cdot n \quad \text{on } \Gamma_2 := (\partial B_2) \cap B_1.
\end{align}

(23) \quad (24) \quad (25) \quad (26)
If $\varphi$ can be extended analytically on the set $B_1 \cap B_2$, we conclude as the previous cases. That is, $\varphi(x) = x \cdot e_1 + \text{const}$, and again $g \in V$, which is a contradiction.

The proof is based on complex analysis (a Mobius transformation and a version of Schwarz reflection principle for harmonic functions).
Setting the problem

Viscous flow

Inviscid flow

\[ \psi_2(z_2) = -\text{Im}[(z_2 + c)^{-1}] \text{ on } d_{-1}, \]
\[ \psi_2(z_2) = \text{Im}[(z_2 + c)^{-1}] \text{ on } d_0, \]
\[ \psi_2(z_2) = \text{Im}[(z_2 + c)^{-1} + 2(e^{-2\theta_2 z_2} + c)^{-1}] \text{ on } d_1, \]
\[ \psi_2(z_2) = \text{Im}[(z_2 + c)^{-1} + 2 \sum_{k=1}^{3} (e^{-2\theta_2 z_2} + c)^{-1}] \text{ on } d_2, \]
\[ \psi_2(z_2) = \text{Im}[(z_2 + c)^{-1} + 2 \sum_{k=1}^{3} (e^{-2\theta_2 z_2} + c)^{-1}] \text{ on } d_3. \]
Numerical reconstruction

\[
\begin{align*}
\Delta \varphi_u &= 0 \quad \text{in } \Omega \setminus B_0 + u \\
\frac{\partial \varphi_u}{\partial n} &= g \quad \text{on } \partial \Omega, \\
\frac{\partial \varphi_u}{\partial n} &= l_u \cdot n \quad \text{on } \partial B_0 + u.
\end{align*}
\]
Let \( \Lambda((h, l); g) = \varphi_m \) be the potential velocity associated with \( B_1(h) \), measured on a part of the external boundary \( \Gamma_m \).

\[
\min_{u \in U_{ad}} J(u) = \min_{u \in U_{ad}} \int_{\Gamma_m} |\varphi_u - \varphi_m|^2 \, ds.
\]
The gradient

\[ J'(h^0, l^0; (h, l)) = \int_{\partial B_0} \left\{ -h \cdot n \frac{\partial^2 \varphi_0}{\partial n^2} + (\nabla \varphi_0 - l^0) \cdot \text{grad}_{\partial \Omega} (h \cdot n) \right\} \psi \, d\sigma \]

\psi \text{ being the unique solution of the adjoint problem}

\[
\begin{cases}
\Delta \psi = 0 & \text{in } \Omega \setminus \overline{B_0}, \\
\frac{\partial \psi}{\partial n} = \varphi_0 - \varphi_m & \text{on } \Gamma_m, \\
\frac{\partial \psi}{\partial n} = 0 & \text{on } \partial B_0 \cup (\partial \Omega \setminus \overline{\Gamma_m}).
\end{cases}
\]
Detecting a moving ball-shaped rigid body

Convergence of the Steepest Descent Method
Perspectives

- Identifiability of the position and velocity of a smooth solid of arbitrary (known) shape (solid different from a disk).
- Identifiability of the shape via measurement over a time interval.
- Numerical detection for an arbitrary (known) shape (ellipses, or regular polygons).
- Numerical reconstruction of the geometry.
Join work with:

- Carlos Conca and Rodrigo Lecaros (DIM-CMM, U de Chile)
- Patricio Cumsille (UBB, Chile).


