The Isoperimetric Problem in the Gauss Space

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Summary

1. Introduction: the geometry in the Gauss space
2. Isoperimetric inequality: existence and uniqueness of the solution
3. Isoperimetric inequality: stability of the solution
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3. Isoperimetric inequality: stability of the solution
Introduction: the geometry in the Gauss space

The Gauss Space is simply $\mathbb{R}^n$ with weight $e^{-|x|^2/2}$, that is for each set $E \subseteq \mathbb{R}^n$ has

$$V_G(E) := \frac{1}{(2\pi)^{n/2}} \int_E e^{-|x|^2/2} dH^n$$

The green part is so that $V_G(\mathbb{R}^n) = 1$

$$P_G(E) := \frac{1}{(2\pi)^{(n-1)/2}} \int_{\partial E} e^{-|x|^2/2} dH^{n-1}$$

$$P(E) := \int_{\partial E} 1 dH^{n-1}$$
Introduction: the geometry in the Gauss space

The Gauss Space is simply $\mathbb{R}^n$ with weight $e^{-|x|^2/2}$, that is for each set $E \subseteq \mathbb{R}^n$ one has

$$V_G(E) := \frac{1}{(2\pi)^{n/2}} \int_E e^{-\frac{|x|^2}{2}} \, d\mathcal{H}^n$$

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Isoperimetric inequality: existence and uniqueness of the solution

What is the isoperimetric set of fixed volume $0 < \alpha < 1$?

- In the Euclidean case, it is well known that the solution is the ball.
- Other densities (non-Gaussian): very badly known.
- The solution is: the half-space (of correct volume).


Isoperimetric inequality: existence and uniqueness of the solution

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Isoperimetric inequality: existence and uniqueness of the solution

What is the isoperimetric set of fixed volume $0 < \alpha < 1$? That is,

$$\min \left\{ P_G(E) : V_G(E) = \alpha \right\}.$$
Isoperimetric inequality: existence and uniqueness of the solution

What is the isoperimetric set of fixed volume $0 < \alpha < 1$? That is,

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Isoperimetric inequality: existence and uniqueness of the solution

What is the *isoperimetric set* of fixed volume $0 < \alpha < 1$? That is,

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What is the \emph{isoperimetric set} of fixed volume $0 < \alpha < 1$? That is,

$$\min \left\{ P_G(E) : V_G(E) = \alpha \right\}.$$

- In the \textbf{Euclidean case}, it is well known that the solution is the \textbf{ball}.
- Other densities (non-Gaussian): very badly known.
- The \textbf{solution} is: the half-space (of correct volume).
Our result

\[ \lambda(E) = \min \{ V(E \Delta B) : B \text{ is a ball}, V(B) = V(E) \} \]
Then
\[ P(E) \geq P(B) + 1 \cdot C \lambda(E)^2. \]

\[ \lambda_G(E) = \min \{ V(E \Delta H) : H \text{ half-space}, V(H) = V(B) \} \]
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Our result


\[ \lambda(E) = \min \{ V(E \triangle B) : B \text{ is a ball}, V(B) = V(E) \} \]

Then \( P(E) \geq P(B) + 1 + C \lambda(E)^2 \).


\[ \lambda_G(E) = \min \{ V(E \triangle H) : H \text{ half-space}, V_G(H) = V_G(B) \} \]

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In N. Fusco–F. Maggi–A.P. (2008) it is proved what follows. Define \( \lambda(E) = \min \{ V(E \Delta B) : B \text{ is a ball, } V(E) = V(B) \} \). Then
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P(E) \geq P(B) + 1 + C \lambda(E)^2.
\]
In A. Cianchi–N. Fusco–F. Maggi–A.P. (2008) it is proved what follows. Define \( \lambda(G(E)) = \min \{ V(G(E) \Delta H) : H \text{ half-space, } V(G(H)) = V(G(B)) \} \). Then
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P_G(E) \geq P_G(H) + 1 + C \lambda(G(E))^2.
\]
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$$P_G(E) \geq P_G(H).$$
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\[
P_G(E) \geq P_G(H) + \frac{1}{C} \lambda_G(E)^2.
\]
The Ehrhard symmetrization: 1D case

Euclidean case: the usual Steiner symmetrization
If \( E \subseteq \mathbb{R} \), then the Steiner symm. of \( E \) is \( E^* := [-\lambda, \lambda] \) with \( V(E) = V(E^*) \).

Gaussian case: the Ehrhard symmetrization
If \( E \subseteq \mathbb{R} \), then the Ehrhard symm. of \( E \) is \( E^* := \left[ \lambda, +\infty \right) \) with \( V_G(E) = V_G(E^*) \).

\[ V(E^*) = V(E) \quad V_G(E^*) = V_G(E) \] (by Fubini)

\[ P(E^*) \leq P(E) \quad P_G(E^*) \leq P_G(E) \] (almost trivially quantitative)
The Ehrhard symmetrization: \(1D\) case

Euclidean case: the usual Steiner symmetrization

\[ E^* := [-\lambda, \lambda] \]

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\[ \text{by Fubini} \]

\[ P(E^*) \leq P(E) \]

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If $E \subseteq \mathbb{R}$, then the Ehrhard symm. of $E$ is

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with $V_G(E) = V_G(E^*)$. 
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If $E \subseteq \mathbb{R}$, then the Ehrhard symm. of $E$ is

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$V(E^*) = V(E)$ \hspace{1cm} (by Fubini)

$P(E^*) \leq P(E)$ \hspace{1cm} (trivially quantitative)
The Ehrhard symmetrization: 1D case

Euclidean case: the usual Steiner symmetrization
If $E \subseteq \mathbb{R}$, then the Steiner symm. of $E$ is

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If $E \subseteq \mathbb{R}$, then the Ehrhard symm. of $E$ is

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$$V(E^*) = V(E) \quad V_G(E^*) = V_G(E) \quad \text{(by Fubini)}$$

$$P(E^*) \leq P(E) \quad P_G(E^*) \leq P_G(E) \quad \text{(almost trivially quantitative)}$$
The Ehrhard symmetrization: general case

For a set \( E \subseteq \mathbb{R}^n \times \mathbb{R}^n - 1_s \times \mathbb{R}^n \), and for each \( s \in \mathbb{R}^n - 1 \), define the section

\[
\mathbb{R} \supseteq E_s := \{ y \in \mathbb{R}^n : (s, y) \in E \}.
\]

Steiner symmetrization: if \( E \subseteq \mathbb{R}^n \), then \( E^* \) is defined by

\[
E^* := E^*_s \cdot E^*.
\]

Ehrhard symmetrization: if \( E \subseteq \mathbb{R}^n \), then \( E^* \) is defined by

\[
E^* := (E_s^* \cdot E^*).
\]

\[
V(E^*) = V(E),
\]

\[
V_G(E^*) = V_G(E),
\]

(by Fubini)

\[
P(E^*) \leq P(E),
\]

\[
P_G(E^*) \leq P_G(E),
\]

(how does one prove this?)
The Ehrhard symmetrization: general case

For a set $E \subseteq \mathbb{R}^n \simeq \mathbb{R}^{n-1} \times \mathbb{R}$, and for each $s \in \mathbb{R}^{n-1}$, define the section

$$\mathbb{R} \ni E_s := \{ y \in \mathbb{R} : (s, y) \in E \}.$$
The Ehrhard symmetrization: general case

For a set $E \subseteq \mathbb{R}_x^n \approx \mathbb{R}_s^{n-1} \times \mathbb{R}_y$, and for each $s \in \mathbb{R}^{n-1}$, define the section

$$\mathbb{R} \ni E_s := \{ y \in \mathbb{R} : (s, y) \in E \}.$$ 

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The Ehrhard symmetrization: general case

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$$\mathbb{R} \supseteq E_s := \{ y \in \mathbb{R} : (s, y) \in E \}.$$ 

Steiner symmetrization: if $E \subseteq \mathbb{R}^n$, then $E^*$ is defined by

$$E_s^* := (E_s)^*.$$
The Ehrhard symmetrization: general case

For a set $E \subseteq \mathbb{R}^n_\times \cong \mathbb{R}^{n-1}_s \times \mathbb{R}_y$, and for each $s \in \mathbb{R}^{n-1}$, define the section

$$\mathbb{R} \supseteq E_s := \{y \in \mathbb{R} : (s, y) \in E\}.$$ 

Steiner symmetrization: if $E \subseteq \mathbb{R}^n$, then $E^*$ is defined by

$$E_s^* := (E_s)^*.$$ 

Ehrhard symmetrization: if $E \subseteq \mathbb{R}^n$, then $E^*$ is defined by
The Ehrhard symmetrization: general case

For a set $E \subseteq \mathbb{R}^n \cong \mathbb{R}_{s}^{n-1} \times \mathbb{R}_{y}$, and for each $s \in \mathbb{R}^{n-1}$, define the section

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Steiner symmetrization: if $E \subseteq \mathbb{R}^n$, then $E^*$ is defined by

$$E_s^* := (E_s)^*.$$ 

Ehrhard symmetrization: if $E \subseteq \mathbb{R}^n$, then $E^*$ is defined by

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$V(E^*) = V(E)$

$V_G(E^*) = V_G(E)$ (by Fubini)

$P(E^*) \leq P(E)$

$P_G(E^*) \leq P_G(E)$ (how does one prove this?)
The Ehrhard symmetrization: general case

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Ehrhard symmetrization: if $E \subseteq \mathbb{R}^n$, then $E^*$ is defined by

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$$V(E^*) = V(E)$$

(by Fubini)

$$P(E^*) \leq P(E)$$

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(how does one prove this?)
The Ehrhard symmetrization: general case

For a set $E \subseteq \mathbb{R}^n \approx \mathbb{R}_{s}^{n-1} \times \mathbb{R}_y$, and for each $s \in \mathbb{R}^{n-1}$, define the section

$$\mathbb{R} \ni E_s := \{ y \in \mathbb{R} : (s, y) \in E \}.$$ 

Steiner symmetrization: if $E \subseteq \mathbb{R}^n$, then $E^*$ is defined by

$$E_s^* := (E_s)^*.$$ 

Ehrhard symmetrization: if $E \subseteq \mathbb{R}^n$, then $E^*$ is defined by

$$E_s^* := (E_s)^*.$$ 

\[V(E^*) = V(E) \quad V_G(E^*) = V_G(E) \quad (\text{by Fubini})\]
\[P(E^*) \leq P(E) \quad P_G(E^*) \leq P_G(E) \quad (\text{how does one prove this?})\]
Consider the functions $v, p: \mathbb{R}^{n-1} \to \mathbb{R}$, defined by

$$v(s) = V(E_s)$$
$$p(s) = P(E_s)$$

One proves that

$$P(E) \geq \int_{\mathbb{R}^{n-1}} \sqrt{|\nabla v(s)|^2 + p(s)^2} \, dH^{n-1}(s)$$

with equality if $E = E^*$. 

As a consequence,

$$P(E) \geq \int_{\mathbb{R}^{n-1}} \sqrt{|\nabla v(s)|^2 + p(s)^2} \, dH^{n-1}(s) \geq \int_{\mathbb{R}^{n-1}} \sqrt{|\nabla v^*(s)|^2 + p^*(s)^2} \, dH^{n-1}(s) = P(E^*)$$

with equality if $E = E^*$. 

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Perimetry in Gauss Space
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How does one prove this (super-sketch)

Consider the functions $v, p : \mathbb{R}^{n-1} \to \mathbb{R}$, $v, p : \mathbb{R}^{n-1} \to \mathbb{R}$, defined by
How does one prove this (super-sketch)

Consider the functions $v, p : \mathbb{R}^{n-1} \rightarrow \mathbb{R}, v, p : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, defined by

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Consider the functions $v, p : \mathbb{R}^{n-1} \to \mathbb{R}$, $v, p : \mathbb{R}^{n-1} \to \mathbb{R}$, defined by

$$v(s) = V(E_s) \quad p(s) = P(E_s)$$

$$v(s) = V_G(E_s) \quad p(s) = P_G(E_s)$$
How does one prove this (super-sketch)

Consider the functions \( v, p : \mathbb{R}^{n-1} \to \mathbb{R}, \) defined by

\[
\begin{align*}
  v(s) &= V(E_s) \\
  p(s) &= P(E_s) \\
  v(s) &= V_G(E_s) \\
  p(s) &= P_G(E_s)
\end{align*}
\]

One proves that

\[
P(E) \geq \int_{\mathbb{R}^{n-1}} \sqrt{|\nabla v(s)|^2 + p(s)^2} \, d\mathcal{H}^{n-1}(s)
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with equality if \( E = E^* \).
Consider the functions \( v, p : \mathbb{R}^{n-1} \to \mathbb{R} \), \( v, p : \mathbb{R}^{n-1} \to \mathbb{R} \), defined by

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v(s) &= V(E_s) \\
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with equality if \( E = E^* \).
How does one prove this (super-sketch)

Consider the functions $v, p : \mathbb{R}^{n-1} \to \mathbb{R}$, $v, p : \mathbb{R}^{n-1} \to \mathbb{R}$, defined by

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  v(s) &= V(E_s) \\
  v(s) &= V_G(E_s) \\
  p(s) &= P(E_s) \\
  p(s) &= P_G(E_s)
\end{align*}
\]

One proofs that

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P_G(E) \geq \int_{\mathbb{R}^{n-1}} \sqrt{\left| \nabla v(s) \right|^2 + p(s)^2} \, d\mathcal{H}^{n-1}(s)
\]

with equality if $E = E^*$. As a consequence,

\[
P(E) \geq \int_{\mathbb{R}^{n-1}} \sqrt{\left| \nabla v(s) \right|^2 + p(s)^2} \, d\mathcal{H}^{n-1}(s)
\]

\[
\geq \int_{\mathbb{R}^{n-1}} \sqrt{\left| \nabla v^*(s) \right|^2 + p^*(s)^2} \, d\mathcal{H}^{n-1}(s) = P(E^*)
\]
How does one prove this (super-sketch)

Consider the functions $v, p : \mathbb{R}^{n-1} \to \mathbb{R}$, defined by

$$
v(s) = \begin{cases} 
V(E_s) & \text{if } v(s) = V_G(E_s) \\
V(E_s) & \text{if } p(s) = P_G(E_s)
\end{cases}$$

$$
p(s) = \begin{cases} 
P(E_s) & \text{if } v(s) = V_G(E_s) \\
P_G(E_s) & \text{if } p(s) = P_G(E_s)
\end{cases}
$$

One proofs that

$$
P_G(E) \geq \int_{\mathbb{R}^{n-1}} \sqrt{|\nabla v(s)|^2 + p(s)^2} \, d\mathcal{H}^{n-1}(s)
$$

with equality if $E = E^*$. As a consequence,

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P_G(E) \geq \int_{\mathbb{R}^{n-1}} \sqrt{|\nabla v(s)|^2 + p(s)^2} \, d\mathcal{H}^{n-1}(s)
\geq \int_{\mathbb{R}^{n-1}} \sqrt{|\nabla v^*(s)|^2 + p^*(s)^2} \, d\mathcal{H}^{n-1}(s) = P(E^*)
$$
The proof of the existence–uniqueness

Step I
There exists an isoperimetric set $E$ (by compactness)

Step II
Almost all the $1$–dimensional sections of $E$ are segments

Step III
The set of points of density $1$ of $E$ is convex

Step IV
Some smart De Giorgi arguments give the thesis

$E$ is convex, $\mathbb{R}^n \setminus E$ is convex, then the thesis follows
The proof of the existence–uniqueness

Step I There exists an isoperimetric set $E$ (by compactness)
The proof of the existence–uniqueness

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

Step II Almost all the 1-dimensional sections of $E$ are segments

Step III (the set of points of density 1 of) $E$ is convex

Step IV some smart De Giorgi arguments give the thesis
The proof of the existence–uniqueness

**Step I** There exists an isoperimetric set $\bar{E}$ (by compactness)

**Step II** Almost all the 1–dimensional sections of $\bar{E}$ are segments
The proof of the existence–uniqueness

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

Step II Almost all the 1–dimensional sections of $\overline{E}$ are segments

Step II Almost all the 1–dimensional sections of $\overline{E}$ are half-lines
The proof of the existence–uniqueness

Step I There exists an isoperimetric set $\bar{E}$ (by compactness)

Step II Almost all the 1–dimensional sections of $\bar{E}$ are half-lines

Step III (the set of points of density 1 of) $\bar{E}$ is convex
The proof of the existence–uniqueness

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

Step II Almost all the 1–dimensional sections of $\overline{E}$ are segments

Step III (the set of points of density 1 of) $\overline{E}$ is convex

Step IV Some smart De Giorgi arguments give the thesis
The proof of the existence–uniqueness

Step I There exists an isoperimetric set \( \overline{E} \) (by compactness)

Step II Almost all the 1–dimensional sections of \( \overline{E} \) are segments

Step III (the set of points of density 1 of) \( \overline{E} \) is convex

Step IV some smart De Giorgi arguments give the thesis
The proof of the existence–uniqueness

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

Step II Almost all the 1–dimensional sections of $\overline{E}$ are segments

Step III (the set of points of density 1 of) $\overline{E}$ is convex

Step IV some smart De Giorgi arguments give the thesis

Step IV $\overline{E}$ is convex, $\mathbb{R}^n \setminus E$ is convex, then the thesis follows
Isoperimetric inequality: stability of the solution

Very different from the Euclidean case

Step A

Make a precise $1$-stability estimate via "surgery".

One finds $P_G(E) \geq P_G(H) + C \lambda(E) \sqrt{\log(1/\lambda(E))}$,

where $H$ is one half-line with same volume as $E$. 

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Perimetry in Gauss Space

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Isoperimetric inequality: stability of the solution

- Very different from the Euclidean case
Isoperimetric inequality: stability of the solution

- Very different from the Euclidean case

**Step A** Make a precise 1D stability estimate via “surgery”.
Very different from the Euclidean case

**Step A** Make a precise 1D stability estimate via “surgery”. One finds

\[ P_G(E) \geq P_G(H) + \frac{1}{C} \lambda(E) \sqrt{\log \left( \frac{1}{\lambda(E)} \right)}, \]

where \( H \) is one half-line with same volume as \( E \).
Step B

Reduce to an \((n-1)\)-dimensional case

What does “reducing” mean? Passing from \(E\) to \(\hat{E}\) so that

\[ \lambda(E) \leq C \lambda(\hat{E}) \]

and

\[ \lambda(E) \leq C' \left( P_G(\hat{E}) - P_G(H) \right)^{1/2} \]

and

\[ \lambda(E) \leq C'' \left( P_G(E) - P_G(H) \right)^{1/2} \]

How do we do? We start from \(E\) and build \(E_1\) and \(E_2\) (drawing!)

Trivially,

\[ P_G(E) \geq P_G(E_1) + P_G(E_2) \]

We are done because either

\[ \lambda(E) \leq C \lambda(E_1), \]

or

\[ \lambda(E) \leq C \lambda(E_2). \]

. . . or we are done anyway! (hand drawing!)

• By induction, we get \(n-1\) symmetries
Step B

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\lambda(E) \leq C' \left( PG(E) - PG(H) \right)^{1/2}
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Step B Reduce to an \((n-1)\)-dimensional case

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Step B

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Step B

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\]

How do we do? We start from \(E\) and build \(E_1\) and \(E_2\) (drawing!)

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• By induction, we get \(n - 1\) symmetries
Step C (in $2D$)

Step C1: A calculation for a particular class of sets

Step C2: An easy but powerful "geometrical" lemma

Step C3: The case when $\sigma$ is big (compared to $\lambda(E)$)

Step C4: The case when $\sigma$ is 0 (smart surgery plus "horizontal pushing")
Step C (in 2D)

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**Step C1** A calculation for a particular class of sets

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Conclusion (in $2D$)

We already know everything if $E = E^*$

Careful comparison between $E$ and $E^*$

Read the paper
Step D We already know everything if $E = E^*$
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- Careful comparison between $E$ and $E^*$
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General result in dimension $n$

- If $E$ is $n$-symmetric, then it is bad, unless $V_G(E) \approx 0$ or $V_G(E) \approx 1$.

- Ehrhard-like symmetrization of codimension 1 → a 2 case!

- Everything can be read by a one-dimensional function (the $(n-1)$-dimensional volume of the sections).

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- Conclusion “directly” follows
Let $E$, $F$ be convex and symmetric w.r.t. the origin. Then is it true that $V_G(E \cap F) \geq V_G(E) V_G(F)$?

Answer: it seems clear (after a while), but there is a proof only in 2D (see Barthe).
Open problems

Let $E, F$ be convex and symmetric w.r.t. the origin. Then is it true that

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