The Isoperimetric Problem in the Gauss Space

Aldo Pratelli

IMDEA Matematicas, Madrid

BCAM, December 5 2008

Summary

1.
2.
3.

Summary

1. Introduction: the geometry in the Gauss space
2.
3.

Summary

1. Introduction: the geometry in the Gauss space
2. Isoperimetric inequality: existence and uniqueness of the solution
3.

Summary

1. Introduction: the geometry in the Gauss space

2. Isoperimetric inequality: existence and uniqueness of the solution

3. Isoperimetric inequality: stability of the solution

Introduction: the geometry in the Gauss space

Introduction: the geometry in the Gauss space

The Gauss Space is simply $\mathbb{R}^n$ with weight $e^{-|x|^2/2}$, that is for each set $E \subseteq \mathbb{R}^n$ on has

$$V_G(E) := \frac{1}{(2\pi)^{n/2}} \int_E e^{-\frac{|x|^2}{2}} \, d\mathscr{H}^n \qquad\qquad V(E) := \int_E 1 \, d\mathscr{H}^n$$

Introduction: the geometry in the Gauss space

The Gauss Space is simply $\mathbb{R}^n$ with weight $e^{-|x|^2/2}$, that is for each set $E \subseteq \mathbb{R}^n$ on has

$$V_G(E) := \frac{1}{(2\pi)^{n/2}} \int_E e^{-\frac{|x|^2}{2}}\, d\mathscr{H}^n \qquad V(E) := \int_E 1\, d\mathscr{H}^n$$

The green part is so that

$$V_G(\mathbb{R}^n) = 1$$

Introduction: the geometry in the Gauss space

The Gauss Space is simply $\mathbb{R}^n$ with weight $e^{-|x|^2/2}$, that is for each set $E \subseteq \mathbb{R}^n$ on has

$$V_G(E) := \frac{1}{(2\pi)^{n/2}} \int_E e^{-\frac{|x|^2}{2}}\, d\mathscr{H}^n \qquad V(E) := \int_E 1\, d\mathscr{H}^n$$

$$P_G(E) := \frac{1}{(2\pi)^{(n-1)/2}} \int_{\partial E} e^{-\frac{|x|^2}{2}}\, d\mathscr{H}^{n-1} \qquad P(E) := \int_{\partial E} 1\, d\mathscr{H}^{n-1}$$

Isoperimetric inequality: existence and uniqueness of the solution

Isoperimetric inequality: existence and uniqueness of the solution

What is the *isoperimetric set* of fixed volume $0 < \alpha < 1$?

Isoperimetric inequality: existence and uniqueness of the solution

What is the *isoperimetric set* of fixed volume $0 < \alpha < 1$? That is,

$$\min \Big\{ P_G(E) : \ V_G(E) = \alpha \Big\} .$$

Isoperimetric inequality: existence and uniqueness of the solution

What is the *isoperimetric set* of fixed volume $0 < \alpha < 1$? That is,

$$\min \Big\{ P_G(E) : \; V_G(E) = \alpha \Big\} .$$

- In the Euclidean case, it is well known that the solution is the ball.

Isoperimetric inequality: existence and uniqueness of the solution

What is the *isoperimetric set* of fixed volume $0 < \alpha < 1$? That is,

$$\min \Big\{ P_G(E) : \; V_G(E) = \alpha \Big\} .$$

- In the Euclidean case, it is well known that the solution is the ball.
- Applications: mainly in Statistics and Calculus of Variations.

Isoperimetric inequality: existence and uniqueness of the solution

What is the *isoperimetric set* of fixed volume $0 < \alpha < 1$? That is,

$$\min \Big\{ P_G(E) : \; V_G(E) = \alpha \Big\} .$$

- In the Euclidean case, it is well known that the solution is the ball.
- Applications: mainly in Statistics and Calculus of Variations.
- Other densities (non-Gaussian): very badly known.

Isoperimetric inequality: existence and uniqueness of the solution

What is the *isoperimetric set* of fixed volume $0 < \alpha < 1$? That is,

$$\min \Big\{ P_G(E) : \; V_G(E) = \alpha \Big\} .$$

- In the Euclidean case, it is well known that the solution is the ball.
- Applications: mainly in Statistics and Calculus of Variations.
- Other densities (non-Gaussian): very badly known.
- The solution is: the half-space (of correct volume).

Isoperimetric inequality: existence and uniqueness of the solution

What is the *isoperimetric set* of fixed volume $0 < \alpha < 1$? That is,

$$\min \Big\{ P_G(E) : \ V_G(E) = \alpha \Big\} .$$

• In the Euclidean case, it is well known that the solution is the ball.
• Applications: mainly in Statistics and Calculus of Variations.
• Other densities (non-Gaussian): very badly known.
• The solution is: the half-space (of correct volume).
• Proof of the "existence": first independently by C. Borell (1975) and V.N. Sudakov–B.S. Cirel'son (1974). Then via "geometric" methods by A. Ehrhard (1983), and via probabilistic methods by D. Bakry–M. Ledoux (1995) and S.G. Bobkov (1997).

Isoperimetric inequality: existence and uniqueness of the solution

What is the *isoperimetric set* of fixed volume $0 < \alpha < 1$? That is,

$$\min \left\{ P_G(E) : \; V_G(E) = \alpha \right\}.$$

• In the Euclidean case, it is well known that the solution is the ball.
• Applications: mainly in Statistics and Calculus of Variations.
• Other densities (non-Gaussian): very badly known.
• The solution is: the half-space (of correct volume).
• Proof of the "existence": first independently by C. Borell (1975) and V.N. Sudakov–B.S. Cirel'son (1974). Then via "geometric" methods by A. Ehrhard (1983), and via probabilistic methods by D. Bakry–M. Ledoux (1995) and S.G. Bobkov (1997).
• Proof of the "uniqueness": E.A. Carlen–C. Kerce (2001) via theory of rearrangements and probabilistic techniques involving the Mehler (Ornstein-Uhlenbeck) semigroup.

Our result

Our result

In N. Fusco–F. Maggi–A.P. (2008) it is proved what follows.

Our result

In N. Fusco–F. Maggi–A.P. (2008) it is proved what follows.
Define $\lambda(E) = \min \left\{ V(E \Delta B) : \ B \text{ is a ball, } V(E) = V(B) \right\}$. Then

Our result

In N. Fusco–F. Maggi–A.P. (2008) it is proved what follows.
Define $\lambda(E) = \min\left\{V(E\Delta B) : \ B \text{ is a ball, } V(E) = V(B)\right\}$. Then

$$P(E) \geq P(B) \qquad .$$

Our result

In N. Fusco–F. Maggi–A.P. (2008) it is proved what follows.
Define $\lambda(E) = \min\left\{V(E\Delta B):\ B \text{ is a ball, } V(E) = V(B)\right\}$. Then

$$P(E) \geq P(B) + \frac{1}{C}\,\lambda(E)^2\,.$$

Our result

In N. Fusco–F. Maggi–A.P. (2008) it is proved what follows.
Define $\lambda(E) = \min \big\{ V(E\Delta B) : \ B \text{ is a ball, } V(E) = V(B) \big\}$. Then

$$P(E) \geq P(B) + \frac{1}{C}\, \lambda(E)^2 \,.$$

In A. Cianchi–N. Fusco–F. Maggi–A.P. (2008) it is proved what follows.

Our result

In N. Fusco–F. Maggi–A.P. (2008) it is proved what follows.
Define $\lambda(E) = \min\left\{V(E\Delta B) : \ B \text{ is a ball, } V(E) = V(B)\right\}$. Then

$$P(E) \geq P(B) + \frac{1}{C}\,\lambda(E)^2\,.$$

In A. Cianchi–N. Fusco–F. Maggi–A.P. (2008) it is proved what follows.
Define $\lambda_G(E) = \min\left\{V(E\Delta H) : \ H \text{ half-space, } V_G(H) = V_G(B)\right\}$. Then

Our result

In N. Fusco–F. Maggi–A.P. (2008) it is proved what follows.
Define $\lambda(E) = \min\left\{V(E\Delta B) : \ B \text{ is a ball, } V(E) = V(B)\right\}$. Then

$$P(E) \geq P(B) + \frac{1}{C}\,\lambda(E)^2 \,.$$

In A. Cianchi–N. Fusco–F. Maggi–A.P. (2008) it is proved what follows.
Define $\lambda_G(E) = \min\left\{V(E\Delta H) : \ H \text{ half-space, } V_G(H) = V_G(B)\right\}$. Then

$$P_G(E) \geq P_G(H) \qquad\qquad .$$

Our result

In N. Fusco–F. Maggi–A.P. (2008) it is proved what follows.
Define $\lambda(E) = \min\left\{V(E\Delta B):\ B \text{ is a ball}, V(E) = V(B)\right\}$. Then

$$P(E) \geq P(B) + \frac{1}{C}\lambda(E)^2 .$$

In A. Cianchi–N. Fusco–F. Maggi–A.P. (2008) it is proved what follows.
Define $\lambda_G(E) = \min\left\{V(E\Delta H):\ H \text{ half-space}, V_G(H) = V_G(B)\right\}$. Then

$$P_G(E) \geq P_G(H) + \frac{1}{C}\lambda_G(E)^2 .$$

The Ehrhard symmetrization: $1D$ case

The Ehrhard symmetrization: $1D$ case

Euclidean case: the usual Steiner symmetrization

The Ehrhard symmetrization: $1D$ case

Euclidean case: the usual Steiner symmetrization
If $E \subseteq \mathbb{R}$, then the Steiner symm. of E is

$$E^* := [-\lambda, \lambda] \qquad \textit{with } V(E) = V(E^*)\,.$$

The Ehrhard symmetrization: $1D$ case

Euclidean case: the usual Steiner symmetrization
If $E \subseteq \mathbb{R}$, then the Steiner symm. of E is

$$E^* := [-\lambda, \lambda] \qquad \textit{with } V(E) = V(E^*)\,.$$

Gaussian case: the Ehrhard symmetrization

The Ehrhard symmetrization: $1D$ case

Euclidean case: the usual Steiner symmetrization
If $E \subseteq \mathbb{R}$, then the Steiner symm. of E is

$$E^* := [-\lambda, \lambda] \qquad \textit{with } V(E) = V(E^*)\,.$$

Gaussian case: the Ehrhard symmetrization
If $E \subseteq \mathbb{R}$, then the Ehrhard symm. of E is

$$E^* := [\lambda, +\infty] \qquad \textit{with } V_G(E) = V_G(E^*)\,.$$

The Ehrhard symmetrization: $1D$ case

Euclidean case: the usual Steiner symmetrization
If $E \subseteq \mathbb{R}$, then the Steiner symm. of E is

$$E^* := [-\lambda, \lambda] \qquad \textit{with } V(E) = V(E^*)\,.$$

Gaussian case: the Ehrhard symmetrization
If $E \subseteq \mathbb{R}$, then the Ehrhard symm. of E is

$$E^* := [\lambda, +\infty] \qquad \textit{with } V_G(E) = V_G(E^*)\,.$$

$V(E^*) = V(E)$ (by Fubini)

$P(E^*) \leq P(E)$ (trivially quantitative)

The Ehrhard symmetrization: $1D$ case

Euclidean case: the usual Steiner symmetrization
If $E \subseteq \mathbb{R}$, then the Steiner symm. of E is

$$E^* := [-\lambda, \lambda] \qquad \text{with } V(E) = V(E^*)\,.$$

Gaussian case: the Ehrhard symmetrization
If $E \subseteq \mathbb{R}$, then the Ehrhard symm. of E is

$$E^* := [\lambda, +\infty] \qquad \text{with } V_G(E) = V_G(E^*)\,.$$

$$V(E^*) = V(E) \qquad V_G(E^*) = V_G(E) \qquad \text{(by Fubini)}$$

$$P(E^*) \leq P(E) \qquad P_G(E^*) \leq P_G(E) \qquad \text{(almost trivially quantitative)}$$

The Ehrhard symmetrization: general case

The Ehrhard symmetrization: general case

For a set $E \subseteq \mathbb{R}^n_x \approx \mathbb{R}^{n-1}_s \times \mathbb{R}_y$, and for each $s \in \mathbb{R}^{n-1}$, define the *section*

$$\mathbb{R} \supseteq E_s := \left\{ y \in \mathbb{R} : \ (s, y) \in E \right\}.$$

The Ehrhard symmetrization: general case

For a set $E \subseteq \mathbb{R}^n_x \approx \mathbb{R}^{n-1}_s \times \mathbb{R}_y$, and for each $s \in \mathbb{R}^{n-1}$, define the *section*

$$\mathbb{R} \supseteq E_s := \left\{ y \in \mathbb{R} : \ (s, y) \in E \right\}.$$

Steiner symmetrization: if $E \subseteq \mathbb{R}^n$, then E^* is defined by

The Ehrhard symmetrization: general case

For a set $E \subseteq \mathbb{R}^n_x \approx \mathbb{R}^{n-1}_s \times \mathbb{R}_y$, and for each $s \in \mathbb{R}^{n-1}$, define the *section*

$$\mathbb{R} \supseteq E_s := \left\{ y \in \mathbb{R} : \ (s, y) \in E \right\}.$$

Steiner symmetrization: if $E \subseteq \mathbb{R}^n$, then E^* is defined by

$$E^*_s := \left(E_s \right)^*.$$

The Ehrhard symmetrization: general case

For a set $E \subseteq \mathbb{R}^n_x \approx \mathbb{R}^{n-1}_s \times \mathbb{R}_y$, and for each $s \in \mathbb{R}^{n-1}$, define the *section*

$$\mathbb{R} \supseteq E_s := \Big\{ y \in \mathbb{R} : \ (s, y) \in E \Big\} .$$

Steiner symmetrization: if $E \subseteq \mathbb{R}^n$, then E^* is defined by

$$E^*_s := \big(E_s \big)^* .$$

Ehrhard symmetrization: if $E \subseteq \mathbb{R}^n$, then E^* is defined by

The Ehrhard symmetrization: general case

For a set $E \subseteq \mathbb{R}^n_x \approx \mathbb{R}^{n-1}_s \times \mathbb{R}_y$, and for each $s \in \mathbb{R}^{n-1}$, define the *section*

$$\mathbb{R} \supseteq E_s := \Big\{ y \in \mathbb{R} : \ (s, y) \in E \Big\} .$$

Steiner symmetrization: if $E \subseteq \mathbb{R}^n$, then E^* is defined by

$$E^*_s := \big(E_s \big)^* .$$

Ehrhard symmetrization: if $E \subseteq \mathbb{R}^n$, then E^* is defined by

$$E^*_s := \big(E_s \big)^* .$$

The Ehrhard symmetrization: general case

For a set $E \subseteq \mathbb{R}^n_x \approx \mathbb{R}^{n-1}_s \times \mathbb{R}_y$, and for each $s \in \mathbb{R}^{n-1}$, define the *section*

$$\mathbb{R} \supseteq E_s := \Big\{ y \in \mathbb{R} : \ (s,y) \in E \Big\} .$$

Steiner symmetrization: if $E \subseteq \mathbb{R}^n$, then E^* is defined by

$$E^*_s := \big(E_s\big)^* .$$

Ehrhard symmetrization: if $E \subseteq \mathbb{R}^n$, then E^* is defined by

$$E^*_s := \big(E_s\big)^* .$$

$V(E^*) = V(E)$ (by Fubini)

$P(E^*) \leq P(E)$ (how does one prove this?)

The Ehrhard symmetrization: general case

For a set $E \subseteq \mathbb{R}^n_x \approx \mathbb{R}^{n-1}_s \times \mathbb{R}_y$, and for each $s \in \mathbb{R}^{n-1}$, define the *section*

$$\mathbb{R} \supseteq E_s := \Big\{ y \in \mathbb{R} : \ (s, y) \in E \Big\} .$$

Steiner symmetrization: if $E \subseteq \mathbb{R}^n$, then E^* is defined by

$$E^*_s := \Big(E_s \Big)^* .$$

Ehrhard symmetrization: if $E \subseteq \mathbb{R}^n$, then E^* is defined by

$$E^*_s := \Big(E_s \Big)^* .$$

$V(E^*) = V(E)$	$V_G(E^*) = V_G(E)$	(by Fubini)
$P(E^*) \leq P(E)$	$P_G(E^*) \leq P_G(E)$	(how does one prove this?)

How does one prove this (super-sketch)

How does one prove this (super-sketch)

Consider the functions $v,\ p : \mathbb{R}^{n-1} \to \mathbb{R}$, $v,\ p : \mathbb{R}^{n-1} \to \mathbb{R}$, defined by

How does one prove this (super-sketch)

Consider the functions $v,\ p : \mathbb{R}^{n-1} \to \mathbb{R}$, $v,\ p : \mathbb{R}^{n-1} \to \mathbb{R}$, defined by

$$v(s) = V(E_s) \qquad\qquad p(s) = P(E_s)$$

How does one prove this (super-sketch)

Consider the functions $v,\ p : \mathbb{R}^{n-1} \to \mathbb{R}$, $v,\ p : \mathbb{R}^{n-1} \to \mathbb{R}$, defined by

$$v(s) = V(E_s) \qquad\qquad p(s) = P(E_s)$$

$$v(s) = V_G(E_s) \qquad\qquad p(s) = P_G(E_s)$$

How does one prove this (super-sketch)

Consider the functions $v,\ p : \mathbb{R}^{n-1} \to \mathbb{R}$, $v,\ p : \mathbb{R}^{n-1} \to \mathbb{R}$, defined by

$$v(s) = V(E_s) \qquad\qquad p(s) = P(E_s)$$

$$v(s) = V_G(E_s) \qquad\qquad p(s) = P_G(E_s)$$

One proofs that

$$P(E) \geq \int_{\mathbb{R}^{n-1}} \sqrt{|\nabla v(s)|^2 + p(s)^2}\, d\mathscr{H}^{n-1}(s)$$

with equality if $E = E^*$.

How does one prove this (super-sketch)

Consider the functions $v,\ p : \mathbb{R}^{n-1} \to \mathbb{R}$, $v,\ p : \mathbb{R}^{n-1} \to \mathbb{R}$, defined by

$$v(s) = V(E_s) \qquad\qquad p(s) = P(E_s)$$

$$v(s) = V_G(E_s) \qquad\qquad p(s) = P_G(E_s)$$

One proofs that

$$P_G(E) \geq \int_{\mathbb{R}^{n-1}} \sqrt{|\nabla v(s)|^2 + p(s)^2}\, d\mathscr{H}^{n-1}(s)$$

with equality if $E = E^*$.

How does one prove this (super-sketch)

Consider the functions $v,\ p : \mathbb{R}^{n-1} \to \mathbb{R}$, $v,\ p : \mathbb{R}^{n-1} \to \mathbb{R}$, defined by

$$v(s) = V(E_s) \qquad p(s) = P(E_s)$$

$$v(s) = V_G(E_s) \qquad p(s) = P_G(E_s)$$

One proofs that

$$P_G(E) \geq \int_{\mathbb{R}^{n-1}} \sqrt{|\nabla v(s)|^2 + p(s)^2}\, d\mathscr{H}^{n-1}(s)$$

with equality if $E = E^*$. As a consequence,

$$\begin{aligned} P(E) &\geq \int_{\mathbb{R}^{n-1}} \sqrt{|\nabla v(s)|^2 + p(s)^2}\, d\mathscr{H}^{n-1}(s) \\ &\geq \int_{\mathbb{R}^{n-1}} \sqrt{|\nabla v^*(s)|^2 + p^*(s)^2}\, d\mathscr{H}^{n-1}(s) = P(E^*) \end{aligned}$$

How does one prove this (super-sketch)

Consider the functions $v,\ p : \mathbb{R}^{n-1} \to \mathbb{R}$, $v,\ p : \mathbb{R}^{n-1} \to \mathbb{R}$, defined by

$$v(s) = V(E_s) \qquad p(s) = P(E_s)$$

$$v(s) = V_G(E_s) \qquad p(s) = P_G(E_s)$$

One proofs that

$$P_G(E) \geq \int_{\mathbb{R}^{n-1}} \sqrt{|\nabla v(s)|^2 + p(s)^2}\, d\mathscr{H}^{n-1}(s)$$

with equality if $E = E^*$. As a consequence,

$$\begin{aligned} P_G(E) &\geq \int_{\mathbb{R}^{n-1}} \sqrt{|\nabla v(s)|^2 + p(s)^2}\, d\mathscr{H}^{n-1}(s) \\ &\geq \int_{\mathbb{R}^{n-1}} \sqrt{|\nabla v^*(s)|^2 + p^*(s)^2}\, d\mathscr{H}^{n-1}(s) = P(E^*) \end{aligned}$$

The proof of the existence–uniqueness

The proof of the existence–uniqueness

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

The proof of the existence–uniqueness

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

The proof of the existence–uniqueness

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

Step II Almost all the $1-$dimensional sections of $\overline{E}$ are segments

The proof of the existence–uniqueness

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

Step II Almost all the 1−dimensional sections of $\overline{E}$ are segments

Step II Almost all the 1−dimensional sections of $\overline{E}$ are half-lines

The proof of the existence–uniqueness

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

Step II Almost all the 1−dimensional sections of $\overline{E}$ are segments

Step II Almost all the 1−dimensional sections of $\overline{E}$ are half-lines

Step III (the set of points of density 1 of) $\overline{E}$ is convex

The proof of the existence–uniqueness

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

Step II Almost all the 1−dimensional sections of $\overline{E}$ are segments

Step II Almost all the 1−dimensional sections of $\overline{E}$ are half-lines

Step III (the set of points of density 1 of) $\overline{E}$ is convex

Step III (the set of points of density 1 of) $\overline{E}$ is convex

The proof of the existence–uniqueness

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

Step II Almost all the 1−dimensional sections of $\overline{E}$ are segments

Step II Almost all the 1−dimensional sections of $\overline{E}$ are half-lines

Step III (the set of points of density 1 of) $\overline{E}$ is convex

Step III (the set of points of density 1 of) $\overline{E}$ is convex

Step IV some smart De Giorgi arguments give the thesis

The proof of the existence–uniqueness

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

Step I There exists an isoperimetric set $\overline{E}$ (by compactness)

Step II Almost all the 1−dimensional sections of $\overline{E}$ are segments

Step II Almost all the 1−dimensional sections of $\overline{E}$ are half-lines

Step III (the set of points of density 1 of) $\overline{E}$ is convex

Step III (the set of points of density 1 of) $\overline{E}$ is convex

Step IV some smart De Giorgi arguments give the thesis

Step IV $\overline{E}$ is convex, $\mathbb{R}^n \setminus E$ is convex, then the thesis follows

Isoperimetric inequality: stability of the solution

Isoperimetric inequality: stability of the solution

- Very different from the Euclidean case

Isoperimetric inequality: stability of the solution

- Very different from the Euclidean case

Step A Make a precise $1D$ stability estimate via “surgery”.

Isoperimetric inequality: stability of the solution

- Very different from the Euclidean case

Step A Make a precise $1D$ stability estimate via “surgery”. One finds

$$P_G(E) \geq P_G(H) + \frac{1}{C}\,\lambda(E)\sqrt{\log\left(\frac{1}{\lambda(E)}\right)}\,,$$

where H is one half-line with same volume as E.

Step B

Step B

Step B Reduce to an $(n-1)$-dimensional case

Step B

Step B Reduce to an $(n-1)$-dimensional case
What does "reducing" mean? Passing from E to $\widehat{E}$ so that

Step B

Step B Reduce to an $(n-1)$-dimensional case
What does "reducing" mean? Passing from E to $\widehat{E}$ so that

$$\lambda(E) \leq C\lambda(\widehat{E}) \quad \textit{and} \quad P_G(\widehat{E}) - P_G(H) \qquad \leq C\,\Big(P_G(E) - P_G(H)\Big)$$

Step B

Step B Reduce to an $(n-1)$-dimensional case
What does "reducing" mean? Passing from E to $\widehat{E}$ so that

$$\lambda(E) \leq C\lambda(\widehat{E}) \quad \textit{and} \quad \Big(P_G(\widehat{E}) - P_G(H)\Big)^{1/2} \leq C\ \Big(P_G(E) - P_G(H)\Big)^{1/2}$$

Step B

Step B Reduce to an $(n-1)$-dimensional case
What does "reducing" mean? Passing from E to $\widehat{E}$ so that

$$\lambda(E) \leq C\lambda(\widehat{E}) \ \leq C'\Big(P_G(\widehat{E}) - P_G(H)\Big)^{1/2} \leq C''\Big(P_G(E) - P_G(H)\Big)^{1/2}$$

Step B

Step B Reduce to an $(n-1)$-dimensional case
What does "reducing" mean? Passing from E to $\widehat{E}$ so that

$$\lambda(E) \leq C\lambda(\widehat{E}) \leq C'\Big(P_G(\widehat{E}) - P_G(H)\Big)^{1/2} \leq C''\Big(P_G(E) - P_G(H)\Big)^{1/2}$$

How do we do? We start from E and build E_1 and E_2 (drawing!)

Step B

Step B Reduce to an $(n-1)$-dimensional case
What does "reducing" mean? Passing from E to $\widehat{E}$ so that

$$\lambda(E) \leq C\lambda(\widehat{E}) \ \leq C'\Big(P_G(\widehat{E}) - P_G(H)\Big)^{1/2} \leq C''\Big(P_G(E) - P_G(H)\Big)^{1/2}$$

How do we do? We start from E and build E_1 and E_2 (drawing!)

Trivially, $P_G(E) \geq \dfrac{P_G(E_1) + P_G(E_2)}{2}$

Step B

Step B Reduce to an $(n-1)$-dimensional case
What does "reducing" mean? Passing from E to $\widehat{E}$ so that

$$\lambda(E) \leq C\lambda(\widehat{E}) \leq C'\Big(P_G(\widehat{E}) - P_G(H)\Big)^{1/2} \leq C''\Big(P_G(E) - P_G(H)\Big)^{1/2}$$

How do we do? We start from E and build E_1 and E_2 (drawing!)

Trivially, $P_G(E) \geq \dfrac{P_G(E_1) + P_G(E_2)}{2}$

We are done because either $\lambda(E) \leq C\lambda(E_1)$, or $\lambda(E) \leq C\lambda(E_2)$...

Step B

Step B Reduce to an $(n-1)$-dimensional case
What does "reducing" mean? Passing from E to $\widehat{E}$ so that

$$\lambda(E) \leq C\lambda(\widehat{E}) \leq C'\Big(P_G(\widehat{E}) - P_G(H)\Big)^{1/2} \leq C''\Big(P_G(E) - P_G(H)\Big)^{1/2}$$

How do we do? We start from E and build E_1 and E_2 (drawing!)

Trivially, $P_G(E) \geq \dfrac{P_G(E_1) + P_G(E_2)}{2}$

We are done because either $\lambda(E) \leq C\lambda(E_1)$, or $\lambda(E) \leq C\lambda(E_2)$...
...or we are done anyway! (hand drawing!)

Step B

Step B Reduce to an $(n-1)$-dimensional case
What does "reducing" mean? Passing from E to $\widehat{E}$ so that

$$\lambda(E) \leq C\lambda(\widehat{E}) \;\leq C'\Big(P_G(\widehat{E}) - P_G(H)\Big)^{1/2} \leq C''\Big(P_G(E) - P_G(H)\Big)^{1/2}$$

How do we do? We start from E and build E_1 and E_2 (drawing!)

Trivially, $P_G(E) \geq \dfrac{P_G(E_1) + P_G(E_2)}{2}$

We are done because either $\lambda(E) \leq C\lambda(E_1)$, or $\lambda(E) \leq C\lambda(E_2)$...
...or we are done anyway! (hand drawing!)
• By induction, we get $n-1$ symmetries

Step C (in $2D$)

Step C (in $2D$)

Step C1 A calculation for a particular class of sets

Step C (in $2D$)

Step C1 A calculation for a particular class of sets

Step C2 An easy but powerful “geometrical” lemma

Step C (in $2D$)

Step C1 A calculation for a particular class of sets

Step C2 An easy but powerful "geometrical" lemma

Step C3 The case when σ is big (compared to $\lambda(E)$)

Step C (in $2D$)

Step C1 A calculation for a particular class of sets

Step C2 An easy but powerful "geometrical" lemma

Step C3 The case when σ is big (compared to $\lambda(E)$)

Step C4 The case when σ is 0 (smart surgery plus "horizontal pushing")

Conclusion (in $2D$)

Conclusion (in $2D$)

Step D We already know everything if $E = E^*$

Conclusion (in $2D$)

Step D We already know everything if $E = E^*$

- Careful comparison between E and E^*

Conclusion (in $2D$)

Step D We already know everything if $E = E^*$

- Careful comparison between E and E^*

- read the paper

General result in dimension n

General result in dimension n

- If E is $n-$symmetric, then it is bad

General result in dimension n

- If E is $n-$symmetric, then it is bad , unless $V_G(E) \approx 0$ or $V_G(E) \approx 1$

General result in dimension n

- If E is $n-$symmetric, then it is bad , unless $V_G(E) \approx 0$ or $V_G(E) \approx 1$

- Ehrhard-like symmetrization of codimension 1 $\longrightarrow$ a $2D$case!

General result in dimension n

• If E is $n-$symmetric, then it is bad , unless $V_G(E) \approx 0$ or $V_G(E) \approx 1$

• Ehrhard-like symmetrization of codimension 1 $\longrightarrow$ a $2D$case!

• Everything can be read by a one-dimensional function (the $(n-1)$-dimensional volume of the sections)

General result in dimension n

- If E is $n-$symmetric, then it is bad , unless $V_G(E) \approx 0$ or $V_G(E) \approx 1$

- Ehrhard-like symmetrization of codimension 1 $\longrightarrow$ a $2D$case!

- Everything can be read by a one-dimensional function (the $(n-1)$-dimensional volume of the sections)

- Conclusion "directly" follows

Open problems

Open problems

Let E, F be convex and symmetric w.r.t. the origin. Then is it true that

$$V_G(E \cap F) \geq V_G(E)V_G(F) \qquad ??$$

Open problems

Let E, F be convex and symmetric w.r.t. the origin. Then is it true that

$$V_G(E \cap F) \geq V_G(E) V_G(F) \qquad ??$$

Answer: it seems clear (after a while), but there is a proof only in $2D$ (see Barthe)