Hardy type inequality
and application to the stability of degenerate stationary waves

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Abstract

This paper is concerned with the asymptotic stability of degenerate stationary waves for viscous conservation laws in the half space. It is proved that the solution converges to the corresponding degenerate stationary wave at the rate $t^{-\alpha/4}$ as $t \to \infty$, provided that the initial perturbation is in the weighted space $L^2_\alpha = L^2(\mathbb{R}_+; (1 + x)^\alpha)$ for $\alpha < \alpha_c(q) := 3 + 2/q$, where $q$ is the degeneracy exponent. This restriction on $\alpha$ is best possible in the sense that the corresponding linearized operator cannot be dissipative in $L^2_\alpha$ for $\alpha > \alpha_c(q)$. Our stability analysis is based on the space-time weighted energy method combined with a Hardy type inequality with the best possible constant.

Keywords: Viscous conservation laws, degenerate stationary waves, asymptotic stability, Hardy inequality.

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1 Introduction

We study the stability problem of degenerate stationary waves for viscous conservation laws in the half space \( x > 0 \):

\[
    u_t + f(u)_x = u_{xx},
\]

\[
    u(0, t) = -1, \quad u(x, 0) = u_0(x). \tag{1.1}
\]

Here the initial function is assumed to satisfy \( u_0(x) \to 0 \) as \( x \to \infty \), and \( f(u) \) is a smooth function of the form

\[
    f(u) = \frac{1}{q}(-u)^{q+1}(1 + g(u)), \quad f''(u) > 0 \text{ for } -1 \leq u < 0, \tag{1.2}
\]

where \( q \) is a positive integer (degeneracy exponent) and \( g(u) = O(|u|) \) for \( u \to 0 \). Since \( f(0) = f'(0) = 0 \) and \( f(u) \) is strictly convex for \(-1 \leq u < 0\), we see that \( f(u) > 0 \) for \(-1 \leq u < 0\). In particular, we have \( 1 + g(u) > 0 \) for \(-1 \leq u \leq 0\). In this situation, the corresponding stationary problem admits a unique solution \( \phi(x) \) (called degenerate stationary wave), which verifies

\[
    \phi_x = f(\phi),
\]

\[
    \phi(0) = -1, \quad \phi(x) \to 0 \text{ as } x \to \infty. \tag{1.3}
\]

We see easily that \( \phi(x) \) behaves like \( \phi(x) \sim -(1 + x)^{-1/q} \). In particular, we have \( \phi(x) = -(1 + x)^{-1/q} \) when \( g(u) \equiv 0 \).

To discuss the stability of the degenerate stationary wave \( \phi(x) \), we introduce the perturbation \( v \) by \( u(x, t) = \phi(x) + v(x, t) \) and rewrite the problem (1.1) as

\[
    v_t + (f(\phi + v) - f(\phi))_x = v_{xx},
\]

\[
    v(0, t) = 0, \quad v(x, 0) = v_0(x), \tag{1.4}
\]

where \( v_0(x) = u_0(x) - \phi(x) \), and \( v_0(x) \to 0 \) as \( x \to \infty \). The stability of degenerate stationary waves was first studied in [15]. It was proved in [15] that if the initial perturbation \( v_0(x) \) is in the weighted space \( L^2_\alpha \), then the perturbation \( v(x, t) \) decays in \( L^2 \) at the rate \( t^{-\alpha/4} \) as \( t \to \infty \), provided that \( \alpha < \alpha_s(q) \), where

\[
    \alpha_s(q) := (q + 1 + \sqrt{3q^2 + 4q + 1})/q.
\]

The decay rate \( t^{-\alpha/4} \) obtained in [15] would be optimal but the restriction \( \alpha < \alpha_s(q) \) was not very sharp. The main purpose of this paper is to relax this
restriction. Indeed, by employing the space-time weighted energy method in [15] and by applying a Hardy type inequality with the best possible constant (see Proposition 2.3), we show the same decay rate \( t^{-\alpha/4} \) under the weaker restriction \( \alpha < \alpha_c(q) := 3+2/q \) (see Theorem 4.1). Notice that \( \alpha_s(q) < \alpha_c(q) \).

It is interesting to note that a similar restriction on the weight is imposed also for the stability of degenerate shock profiles (see [10]). We remark that our stability result for degenerate stationary waves is completely different from those for non-degenerate case. In fact, for non-degenerate stationary waves, we have the better decay rate \( t^{-\alpha/2} \) for the perturbation without any restriction on \( \alpha \). See [4, 5, 6, 14, 16] for the details. See also [2, 7, 9, 11] for the related stability results for stationary waves.

In this paper we also discuss the dissipativity of the following linearized operator associated with (1.4):

\[
Lv = v_{xx} - (f'(\phi)v)_x.
\]  

(1.5)

In a simpler situation including the case \( g(u) \equiv 0 \) in (1.2), we show that the operator \( L \) is uniformly dissipative in \( L^2_\alpha \) for \( \alpha < \alpha_c(q) \) but cannot be dissipative for \( \alpha > \alpha_c(q) \) (see Theorem 3.5). This suggests that the exponent \( \alpha_c(q) \) is the critical exponent of the stability problem of degenerate stationary waves. This result on the characterization of the dissipativity of \( L \) is an improvement on the previous one in [15] and is established by using a Hardy type inequality with the best possible constant (see Proposition 2.3).

The contents of this paper are as follows. In Section 2 we introduce several Hardy type inequalities and discuss the best possibility of their constants. In Section 3 we discuss the dissipativity of the operator \( L \) in (1.5) in weighted \( L^2 \) spaces. Finally in Section 4, we study the nonlinear stability of degenerate stationary waves.

**Notations.** For \( 1 \leq p \leq \infty \), \( L^p = L^p(\mathbb{R}_+) \) denotes the usual Lebesgue space on \( \mathbb{R}_+ = (0, \infty) \) with the norm \( \| \cdot \|_{L^p} \). Let \( s \) be a nonnegative integer. Then the Sobolev space \( W^{s,p} = W^{s,p}(\mathbb{R}_+) \) is defined by \( W^{s,p} = \{ u \in L^p; \partial^k_x u \in L^p \text{ for } k \leq s \} \) with the norm \( \| \cdot \|_{W^{s,p}} \). When \( p = 2 \), we write \( H^s = W^{s,2} \).

Next we introduce weighted spaces. Let \( w = w(x) > 0 \) be a weight function defined on \( [0, \infty) \) such that \( w \in C^0(0, \infty) \). Then, for \( 1 \leq p < \infty \), we denote by \( L^p(w) \) the weighted \( L^p \) space on \( \mathbb{R}_+ \) equipped with the norm

\[
\| u \|_{L^p(w)} := \left( \int_0^\infty |u(x)|^p w(x) \, dx \right)^{1/p}.
\]  

(1.6)

The corresponding weighted Sobolev space \( W^{s,p}(w) \) is defined by \( W^{s,p}(w) = \{ u \in L^p(w); \partial^k_x u \in L^p(w) \text{ for } k \leq s \} \) with the norm \( \| \cdot \|_{W^{s,p}(w)} \). Also, we
denote by $W^{1,p}_0(w)$ the completion of $C_0^\infty(\mathbb{R}_+)$ with respect to the norm
\[ \|u\|_{W^{1,p}_0(w)} := \|\partial_x u\|_{L^p(w)} = \left( \int_0^\infty |\partial_x u(x)|^p w(x) \, dx \right)^{1/p}. \] (1.7)

When $p = 2$, we write $H^s(w) = W^{s,2}_0(w)$ and $H^1_0(w) = W^{1,2}_0(w)$. In the special case where $w = (1 + x)^\alpha$ with $\alpha \in \mathbb{R}$, these weighted spaces are abbreviated as
\[ L^p_\alpha = L^p((1 + x)^\alpha), \quad W^{s,p}_\alpha = W^{s,p}((1 + x)^\alpha), \quad W^{1,p}_\alpha = W^{1,p}_0((1 + x)^\alpha), \quad H^s_\alpha = H^s((1 + x)^\alpha), \quad H^1_{1,0} = H^1_0((1 + x)^\alpha). \]

Let $k$ be a nonnegative integer. Then, for an interval $I \subset [0, \infty)$ and a Banach space $X$ on $\mathbb{R}_+$, $C^k(I; X)$ denotes the space of $k$-times continuously differentiable functions on $I$ with values in $X$.

Finally, letters $C$ and $c$ in this paper denote positive generic constants which may vary from line to line.

## 2 Hardy type inequality

Hardy’s inequality was first introduced by Hardy [1] and its best possible constant was given by Landau [8]. Here we introduce several Hardy type inequalities which will be used in this paper. The following first one is found in [13] but its best possible constant is not given explicitly there.

**Proposition 2.1.** Let $\psi \in C^1[0, \infty)$ and assume either

(1) $\psi > 0$, $\psi_x > 0$ and $\psi(x) \to \infty$ for $x \to \infty$; or

(2) $\psi < 0$, $\psi_x > 0$ and $\psi(x) \to 0$ for $x \to \infty$.

Then we have
\[ \int_0^\infty v^2 \psi_x \, dx \leq 4 \int_0^\infty v_x^2 \psi \psi_x \, dx \] (2.1)
for $v \in C_0^\infty(\mathbb{R}_+)$ and hence for $v \in H^1_0(w)$ with $w = \psi^2/\psi_x$. Here 4 is the best possible constant, and there is no function $v \in H^1_0(w)$, $v \neq 0$, which attains the equality in (2.1).

**Proof.** Let $v \in C_0^\infty(\mathbb{R}_+)$. A simple calculation gives
\[ (v^2 \psi)_x = v^2 \psi_x + 2vv_x \psi \]
\[ = \frac{1}{2} v^2 \psi_x + \frac{1}{2} \left( v + 2v_x \psi/\psi_x \right)^2 \psi_x - 2v_x^2 \psi^2/\psi_x. \] (2.2)
Integrating (2.2) in $x$, we obtain
\[
\int_0^\infty v^2 \psi_x \, dx + \int_0^\infty (v + 2v_x \psi/\psi_x)^2 \, dx = 4 \int_0^\infty v_x^2 \psi^2/\psi_x \, dx, \tag{2.3}
\]
which gives the desired inequality (2.1).

It follows from (2.3) that the equality in (2.1) holds if and only if $v + 2v_x \psi/\psi_x \equiv 0$. This gives $v = C_1 |\psi|^{-1/2}$ for some constant $C_1$. But, if $C_1 \neq 0$, this $v$ is not in $H_0^1(w)$ with $w = \psi^2/\psi_x$. In fact, in the case (1), we have
\[
v_x = -\frac{1}{2} C_1 \psi^{-3/2} \psi_x \quad \text{and hence}
\int_0^\infty v_x^2w \, dx = \frac{1}{4} C_1^2 \int_0^\infty \psi^{-1} \psi_x \, dx = \frac{1}{4} C_1^2 \left[ \log \psi(x) \right]_0^\infty = \infty.
\]
The case (2) can be treated similarly. Thus we conclude that there is no function $v \in H_0^1(w)$, $v \neq 0$, which attains the equality in (2.1).

Finally, we show the best possibility of the constant 4 in (2.1). The following proof is based on the computation in [8]. First, we consider the case (1). Let us fix $a > 0$. Let $\epsilon > 0$ be a small parameter and put
\[
v'\bigl(x\bigr) = \begin{cases} 0, & 0 \leq x < a, \\ (x-a)\psi\bigl(x\bigr)^{-1/2-\epsilon}, & a < x < a+1, \\ \psi(\bigl(x\bigr)^{-1/2-\epsilon}, & a+1 < x. \end{cases} \tag{2.4}
\]
Then we have
\[
\int_0^\infty (v')^2 \psi_x \, dx = \int_a^{a+1} (x-a)^2 \psi^{-1-2\epsilon} \psi_x \, dx + \int_{a+1}^\infty \psi^{-1-2\epsilon} \psi_x \, dx \\
=: I_1 + I_2.
\]
Here we see that $I_1 = O(1)$ for $\epsilon \to 0$ and
\[
I_2 = \left[ -\frac{1}{2\epsilon} \psi(\bigl(x\bigr)^{-2\epsilon} \right]_{x=a+1}^\infty = \frac{1}{2\epsilon} \psi(a+1)^{-2\epsilon}.
\]
On the other hand, we have
\[
v_x'(x) = \begin{cases} 0, & 0 \leq x < a, \\ \psi(\bigl(x\bigr)^{-1/2-\epsilon} - (1/2 + \epsilon)(x-a)\psi(\bigl(x\bigr)^{-3/2-\epsilon} \psi_x(\bigl(x\bigr), & a < x < a+1, \\ -\psi(\bigl(x\bigr)^{-3/2-\epsilon} \psi_x(\bigl(x\bigr), & a+1 < x.
\]
Therefore we get
\[
\int_0^\infty (v'_x)^2 \psi_x^2 \, dx \\
= \int_a^{a+1} \left\{ \psi^{-1-\epsilon} - (1/2 + \epsilon)(x-a)\psi^{-3/2-\epsilon} \psi_x \right\}^2 \psi_x^2 \, dx \\
+ (1/2 + \epsilon)^2 \int_{a+1}^\infty \psi^{-1-2\epsilon} \psi_x \, dx =: J_1 + J_2.
\]
Here we find that \( J_1 = O(1) \) for \( \epsilon \to 0 \) and \( J_2 = (1/2 + \epsilon)^2 \frac{1}{2\epsilon} \psi(a + 1)^{-2\epsilon} \). Consequently, we obtain

\[
\frac{\int_0^\infty (v')^2 \psi^2 / \psi_x \, dx}{\int_0^\infty (v')^2 \psi_x \, dx} = \frac{O(1) + (1/2 + \epsilon)^2 \frac{1}{2\epsilon} \psi(a + 1)^{-2\epsilon}}{O(1) + \psi(a + 1)^{-2\epsilon}} \xrightarrow{\epsilon \to 0} \frac{1}{4}
\]

for \( \epsilon \to 0 \). This shows that 4 in (2.1) is the best possible constant.

In the case (2), to show the best possibility of the constant in (2.1), we may take a test function \( v(x) \) as

\[
v(x) = \begin{cases} 
0, & 0 \leq x < a, \\
(x - a)(-\psi(x))^{-1/2 - \epsilon}, & a < x < a + 1, \\
(-\psi(x))^{-1/2 - \epsilon}, & a + 1 < x,
\end{cases}
\]

but we omit the details. This completes the proof of Proposition 2.1.

We have the \( L^p \) version of Proposition 2.1.

**Proposition 2.2.** Let \( \psi \) be the same as in Proposition 2.1. Let \( 1 < p < \infty \). Then we have

\[
\int_0^\infty |v|^p \psi_x \, dx \leq p^p \int_0^\infty |v_x|^p |\psi|^p / \psi_x^{p-1} \, dx
\]

for \( v \in C_0^\infty(\mathbb{R}_+) \) and hence for \( v \in W_0^{1,p}(w) \) with \( w = |\psi|^p / \psi_x^{p-1} \). Here \( p^p \) is the best possible constant, and there is no function \( v \in W_0^{1,p}(w) \), \( v \neq 0 \), which attains the equality in (2.5).

**Proof.** Let \( 1 < p < \infty \) and \( v \in C_0^\infty(\mathbb{R}_+) \). A simple calculation gives

\[
(|v|^p \psi)_x = |v|^p \psi_x + p |v|^{p-2} vv_x \psi = \frac{1}{p} (|v|^p \psi_x - p^p |v_x|^p |\psi|^p / \psi_x^{p-1}) + R,
\]

where

\[
R = \left(1 - \frac{1}{p}\right)|v|^p \psi_x + \frac{1}{p} p^p |v_x|^p |\psi|^p / \psi_x^{p-1} + p |v|^{p-2} vv_x \psi.
\]

Integrating (2.6) in \( x \), we obtain

\[
\int_0^\infty |v|^p \psi_x \, dx + p \int_0^\infty R \, dx = p^p \int_0^\infty |v_x|^p |\psi|^{p-1} \psi_x \, dx.
\]
Here we see that
\[-p|v|^{p-2}v v_x \psi \leq p|v|^{p-1}|v_x||\psi| = (|v|^{p-1}v_x^{(p-1)/p})(p|v_x||\psi|/\psi_x^{(p-1)/p}) \leq (1 - 1/p)|v|^p\psi_x + 1/p p^p|v|^p|\psi|/\psi_x^{p-1},\]
where we have used the Young inequality \( AB \leq (1 - 1/p)A^{p/(p-1)} + (1/p)B^p \) for \( A = |v|^{p-1}\psi_x^{(p-1)/p} \) and \( B = p|v_x||\psi|/\psi_x^{(p-1)/p} \). Thus we have \( R \geq 0 \). This together with (2.7) gives the desired inequality (2.5).

It follows from (2.7) that the equality in (2.5) holds if and only if \( R \equiv 0 \). This is the case where \( \psi v \leq 0 \) and \( |v|^p\psi_x = p|v_x||\psi|/\psi_x^{p-1} \). This is equivalent to \( p\psi v \equiv -v\psi_x \) and hence we have \( v = C_1|\psi|^{-1/p} \) for some constant \( C_1 \). A simple computation shows that when \( C_1 \neq 0 \), this \( v \) is not in \( W_0^1(p,w) \) with \( w = |\psi|/\psi_x^{p-1} \). Thus we conclude that there is no function \( v \in W_0^1(p,w) \), \( v \neq 0 \), which attains the equality in (2.5).

The best possibility of the constant \( p^p \) is proved in the same way as in the proof of Proposition 2.1. For example, in the case (1), we take the test function as
\[
v^0(x) = \begin{cases} 0, & 0 \leq x < a, \\ (x-a)|\psi|^p(x)^{-1/p-\epsilon}, & a < x < a + 1, \\ |\psi|^p(x)^{-1/p-\epsilon}, & a + 1 < x. \end{cases} \quad (2.8)
\]
where \( a > 0 \) is fixed and \( \epsilon > 0 \) is a small parameter. Then we see that
\[
\frac{\int_0^\infty |v^0|^p |\psi|/\psi_x^{p-1} \, dx}{\int_0^\infty |v^0|^p \psi_x \, dx} = \frac{O(1) + (1/p + \epsilon)p^{1/p} \psi(a + 1)^{-p\epsilon}}{O(1) + \frac{1}{p^2} \psi(a + 1)^{-p\epsilon}} = \frac{O(\epsilon) + (1/p + \epsilon)p \psi(a + 1)^{-p\epsilon}}{O(\epsilon) + \psi(a + 1)^{-p\epsilon}} \to \frac{1}{p^p}
\]
for \( \epsilon \to 0 \). This shows that \( p^p \) in (2.5) is the best possible constant. Thus the proof of Proposition 2.2 is complete.

The following variant of Proposition 2.1 is useful in our application.

**Proposition 2.3.** Let \( \phi \in C^1[0,\infty) \), \( \phi < 0 \), \( \phi_x > 0 \), and \( \phi(x) \to 0 \) for \( x \to \infty \). Let \( \sigma \in \mathbb{R} \) with \( \sigma \neq 0 \), and define the weight functions \( w \) and \( w_1 \) by
\[
w = (-\phi)^{-\sigma + 1}/\phi_x, \quad w_1 = (-\phi)^{-\sigma - 1}\phi_x. \quad (2.9)
\]
Then we have
\[
\int_0^\infty v^2 w_1 \, dx \leq \frac{4}{\sigma^2} \int_0^\infty v_x^2 w \, dx \quad (2.10)
\]
for $v \in H^1_0(w)$. Here $4/\sigma^2$ is the best possible constant, and there is no function $v \in H^1_0(w), v \neq 0$, which attains the equality in (2.10).

**Proof.** Let $\sigma > 0$. In this case we put $\psi = (-\phi)^{-\sigma} > 0$. Then we have $\psi_x = \sigma(-\phi)^{-\sigma - 1}\phi_x > 0$ and $\psi(x) \to \infty$ as $x \to \infty$. This corresponds to the case (1) of Proposition 2.1. Since $\psi^2/\psi_x = (1/\sigma)(-\phi)^{-\sigma + 1}/\phi_x$, by applying Proposition 2.1, we have

$$\sigma \int_0^\infty \psi^2(-\phi)^{-\sigma - 1}\phi_x \, dx \leq \frac{4}{\sigma} \int_0^\infty \psi_x^2(-\phi)^{-\sigma + 1}/\phi_x \, dx.$$ 

This gives (2.10) and hence the proof of Proposition 2.3 is complete for $\sigma > 0$.

When $\sigma < 0$, we put $\psi = -(\phi)^{-\sigma} < 0$. Then, applying the case (2) of Proposition 2.1, we get the desired conclusion also for $\sigma < 0$. This completes the proof of Proposition 2.3. \qed

As a simple corollary of Proposition 2.3, we have:

**Corollary 2.4.** Let $\alpha \in \mathbb{R}$ with $\alpha \neq 1$. Then we have

$$\|v\|_{L^2_{\alpha-2}}^2 \leq \frac{2}{|\alpha - 1|} \|v_x\|_{L^2_{\alpha}}^2$$

(2.11)

for $v \in H^1_{\alpha,0} = H^1_0((1 + x)^\alpha)$. Here the constant $2/|\alpha - 1|$ is the best possible, and there is no function $v \in H^1_{\alpha,0}, v \neq 0$, which attains the equality in (2.11).

**Proof.** Let $\phi = -(1 + x)^{-1/q}$ with $q > 0$. Then we see that $\phi < 0, \phi_x = (1/q)(1 + x)^{-1/q-1} = (1/q)(-\phi)^{q+1} > 0$, and $\phi(x) \to 0$ as $x \to \infty$. Now we apply Proposition 2.3. Since $w = (-\phi)^{-\sigma + 1}/\phi_x = q(-\phi)^{-\sigma - q} = q(1 + x)^{\sigma/q+1}$, $w_1 = (-\phi)^{-\sigma - 1}\phi_x = \frac{1}{q}(-\phi)^{-\sigma + q} = \frac{1}{q}(1 + x)^{\sigma/q-1}$, (2.12)

we have from (2.10) that

$$\frac{1}{q} \int_0^\infty v^2(1 + x)^{\sigma/q-1} \, dx \leq \frac{4q}{\sigma^2} \int_0^\infty v_x^2(1 + x)^{\sigma/q+1} \, dx$$

for $v \in H^1_{\sigma/q+1,0}$. Thus we have

$$\|v\|_{L^2_{\alpha/q-1}}^2 \leq \frac{4q^2}{\sigma^2} \|v_x\|_{L^2_{\alpha/q+1}}^2,$$

(2.13)

for $v \in H^1_{\sigma/q+1,0}$. This together with $\sigma = (\alpha - 1)q$ gives the desired inequality (2.11). This completes the proof. \qed
3 Dissipativity of the linearized operator

We discuss the dissipativity of the operator $L$ defined by (1.5) in the weighted space $L^2(w)$. For this purpose, we first review the basic properties of the degenerate stationary wave $\phi(x)$ (see [9] for the details).

Lemma 3.1. Suppose that $f(u)$ satisfies (1.2). Then the stationary wave $\phi(x)$, which is a solution of (1.3), verifies the following properties: $\phi \in C^\infty[0, \infty)$, and

$$-1 \leq \phi(x) < 0, \quad \phi_x(x) > 0, \quad \phi(x) \to 0 \text{ for } x \to \infty. \quad (3.1)$$

Moreover, we have

$$c(1+x)^{-1/q} \leq -\phi(x) \leq C(1+x)^{-1/q}. \quad (3.2)$$

Now, let $w > 0$ be a weight function depending only on $x$ such that $w \in C^2[0, \infty)$ and we calculate the inner product $\langle Lv, v \rangle_{L^2(w)}$ for $v \in C^\infty(\mathbb{R}_+)$, where

$$\langle u, v \rangle_{L^2(w)} := \int_0^\infty uvw \, dx. \quad (3.3)$$

We multiply (1.5) by $v$. Then a simple computation gives

$$(Lv)v = \left( vv_x - \frac{1}{2} f'(\phi)v^2 \right)_x - v^2_x - \frac{1}{2} f''(\phi)\phi_x v^2. \quad (3.4)$$

Multiplying by $w$, we obtain

$$(Lv)v w = \left\{ \left( vv_x - \frac{1}{2} f'(\phi)v^2 \right)w - \frac{1}{2} v^2 w_x \right\}_x$$

$$- v^2_x w + \frac{1}{2} v^2(w_{xx} + w_x f'(\phi) - w f''(\phi)\phi_x).$$

Now we choose the weight function $w$ and the corresponding $w_1$ in terms of the degenerate stationary wave $\phi$ by (2.9), where $\sigma \in \mathbb{R}$. Then we have $w = (-\phi)^{-\sigma+1}/f(\phi)$ by $\phi_x = f(\phi)$. Differentiating this expression with respect to $x$ and using $\phi_x = f(\phi)$ several times, we find by direct computations that

$$w_x = (\sigma - 1)(-\phi)^{-\sigma} - (-\phi)^{-\sigma+1} f'(\phi)/f(\phi),$$

$$w_{xx} = \sigma(\sigma - 1)(-\phi)^{-\sigma-1} f(\phi) - (\sigma - 1)(-\phi)^{-\sigma} f'(\phi)$$

$$- (-\phi)^{-\sigma+1} (f''(\phi) - f'(\phi)^2/f(\phi)).$$
Consequently, we arrive at the expression
\[
\begin{align*}
w_{xx} + w_x f'(\phi) - w f''(\phi) \phi_x &= \sigma(\sigma - 1)(-\phi)^{-\sigma -1} f(\phi) - 2(-\phi)^{-\sigma -1} f''(\phi) \\
&= \{\sigma(\sigma - 1) - 2(-\phi)^2 f''(\phi)/f(\phi)\}(-\phi)^{-\sigma -1} f(\phi) \\
&= 2(c_1(\sigma) - r(\phi)) w_1,
\end{align*}
\]
where \(w_1\) is given in (2.9) and
\[
c_1(\sigma) := \sigma(\sigma - 1)/2 - q(q + 1),
\]
\[
r(u) := (-u)^2 f''(u)/f(u) - q(q + 1).
\]

Substituting (3.5) into (3.4) and integrating with respect to \(x\), we get the following conclusion.

**Claim 3.2.** Let \(\phi\) be the degenerate stationary wave and define the weight functions \(w\) and \(w_1\) by (2.9) with \(\sigma \in \mathbb{R}\). Then the operator \(L\) defined in (1.5) verifies
\[
\langle Lv, v \rangle_{L^2(w)} = -\|v_x\|_{L^2(w)}^2 + c_1(\sigma)\|v\|_{L^2(w_1)}^2 - \int_0^\infty v^2 r(\phi) w_1 \, dx
\]
for \(v \in C_0^\infty(\mathbb{R}_+)\) and hence for \(v \in H_0^1(w)\), where \(c_1(\sigma)\) and \(r(\phi)\) are given in (3.6).

To discuss the dissipativity of \(L\), we need to estimate the term \(r(\phi)\) in (3.7). By straightforward computations, using (3.6) and (1.2), we see that
\[
r(u) = (-u)^2 g''(u)/g(u) - 2(q + 1) g'(u)/(1 + g(u)).
\]
This shows that \(r(u) = O(|u|)\) for \(u \to 0\). In particular, we have \(r(u) \equiv 0\) if \(g(u) \equiv 0\). With these preparations, we have the following result on the characterization of the dissipativity of \(L\).

**Theorem 3.3.** Assume (1.2). Let \(\phi\) be the degenerate stationary wave and \(L\) be the operator defined in (1.5). Let \(w\) and \(w_1\) be the weight functions in (2.9) with the parameter \(\sigma \in \mathbb{R}\). Then we have:

1. Let \(-2q < \sigma < 2(q + 1)\). Then, under the additional assumption that \(r(u) \geq 0\) for \(-1 \leq u \leq 0\), the operator \(L\) is uniformly dissipative in \(L^2(w)\). Namely, there is a positive constant \(\delta\) such that
\[
\langle Lv, v \rangle_{L^2(w)} \leq -\delta(\|v_x\|_{L^2(w)}^2 + \|v\|_{L^2(w_1)}^2)
\]
for \(v \in H_0^1(w)\).
Remark 3.4. In (1) of this theorem, we have assumed that \( \eta \leq 0 \) in (1.2) and for \( c_q^2(\sigma > 1) \) be treated similarly and we omit the argument in this latter case. When \( v \) we can choose nonnegative integer \( \sigma \). The simplest example of such a \( g(u) \) is \( g(u) = (-u)^m \) with a nonnegative integer \( m \).

Proof. The proof is based on the equality (3.7) in Claim 3.2 and the Hardy type inequality (2.10) in Proposition 2.3.

Let \( -2q < \sigma < 2(q + 1) \). This is equivalent to \( c_1(\sigma) < \sigma^2/4 \). Therefore we can choose \( \delta > 0 \) so small that \( \delta (1 + \sigma^2/4) \leq \sigma^2/4 - c_1(\sigma) \). Since \( r(\phi) \geq 0 \) by the additional assumption on \( r(u) \) and since \( (\sigma^2/4) \| v \|^2_{L^2(w)} \leq \| v \|^2_{L^2(w)} \) by the Hardy type inequality (2.10), we have from (3.7) that

\[
\langle L v, v \rangle_{L^2(w)} \leq -\| v_x \|^2_{L^2(w)} + c_1(\sigma) \| v \|^2_{L^2(w)} \\
= -\delta \| v_x \|^2_{L^2(w)} - (1 - \delta) \| v_x \|^2_{L^2(w)} + c_1(\sigma) \| v \|^2_{L^2(w)} \\
\leq -\delta \| v_x \|^2_{L^2(w)} - \{(1 - \delta) \sigma^2/4 - c_1(\sigma)\} \| v \|^2_{L^2(w)} \\
\leq -\delta \| v_x \|^2_{L^2(w)} + \| v \|^2_{L^2(w)}
\]

for \( v \in C_0^\infty(\mathbb{R}_+) \) and hence for \( v \in H^1_0(w) \), where we used the fact that \( (1 - \delta) \sigma^2/4 - c_1(\sigma) \geq \delta \). This completes the proof of the uniform dissipative case (1).

Next we consider the case where \( \sigma > 2(q + 1) \); the case \( \sigma < -2q \) can be treated similarly and we omit the argument in this latter case. When \( \sigma > 2(q + 1) \), we have \( c_1(\sigma) > \sigma^2/4 \). Then we choose \( \delta > 0 \) so small that \( c_1(\sigma) \geq \sigma^2/4 + 3\delta \). Since \( r(u) = O(|u|) \) for \( u \to 0 \) and \( \phi(x) \to 0 \) for \( x \to \infty \), we take \( a = a(\delta) > 0 \) so large that \( |r(\phi)| \leq \delta \) for \( x \geq a \). For this choice of \( a \) and for \( \epsilon > 0 \), we take a test function \( v^\epsilon \) as in (2.4):

\[
v^\epsilon(x) = \begin{cases} 0, & 0 \leq x < a, \\
(x - a)(-\phi(x))^{(1/2+\epsilon)}, & a < x < a + 1, \\
(-\phi(x))^{(1/2+\epsilon)}, & a + 1 < x. \end{cases}
\]

Then we have

\[
\left| \int_0^\infty (v^\epsilon)^2 r(\phi) w_1 \, dx \right| \leq \delta \int_a^\infty (v^\epsilon)^2 w_1 \, dx = \delta \| v^\epsilon \|^2_{L^2(w)}.
\]
so that we have from (3.7) that
\[ \langle Lv^\epsilon, v^\epsilon \rangle_{L^2(w)} \geq -\|v_x^\epsilon\|^2_{L^2(w)} + (c_1(\sigma) - \delta)\|v^\epsilon\|^2_{L^2(w_1)}. \] (3.11)

Here, a direct computation shows that
\begin{align*}
\|v^\epsilon\|^2_{L^2(w_1)} &= \int_a^{a+1} \frac{(x - a)^2}{2 \sigma \epsilon} (\phi(a + 1))^{2\sigma \epsilon} dx + \int_{a+1}^{\infty} \frac{(-\phi)^{2\sigma \epsilon} - 1}{2 \sigma \epsilon} \phi_x dx \\
&= O(1) + \frac{1}{2 \sigma \epsilon} (-\phi(a + 1))^{2\sigma \epsilon}
\end{align*}

for \( \epsilon \to 0 \), where the term denoted by \( O(1) \) depends on \( \delta \). Similarly, we have
\begin{align*}
\|v_x^\epsilon\|^2_{L^2(w_1)} &= O(1) + \sigma^2 (1/2 + \epsilon)^2 \frac{1}{2 \sigma \epsilon} (-\phi(a + 1))^{2\sigma \epsilon}
\end{align*}

for \( \epsilon \to 0 \). Consequently, we obtain
\begin{align*}
\frac{\|v^\epsilon\|^2_{L^2(w)}}{\|v^\epsilon\|^2_{L^2(w_1)}} &= \frac{O(1) + \sigma^2 (1/2 + \epsilon)^2 \frac{1}{2 \sigma \epsilon} (-\phi(a + 1))^{2\sigma \epsilon}}{O(1) + \frac{1}{2 \sigma \epsilon} (-\phi(a + 1))^{2\sigma \epsilon}} \\
&= \frac{O(\epsilon) + \sigma^2 (1/2 + \epsilon)^2 (-\phi(a + 1))^{2\sigma \epsilon}}{O(\epsilon) + (-\phi(a + 1))^{2\sigma \epsilon}} \to \frac{\sigma^2}{4}
\end{align*}

for \( \epsilon \to 0 \). Thus we have \( \|v_x^\epsilon\|^2_{L^2(w)} / \|v^\epsilon\|^2_{L^2(w_1)} \leq \sigma^2 / 4 + \delta \) for a suitably small \( \epsilon = \epsilon(\delta) > 0 \). Consequently, we have from (3.11) that
\begin{align*}
\frac{\langle Lv^\epsilon, v^\epsilon \rangle_{L^2(w)}}{\|v^\epsilon\|^2_{L^2(w_1)}} &\geq -\frac{\|v_x^\epsilon\|^2_{L^2(w)}}{\|v^\epsilon\|^2_{L^2(w_1)}} + c_1(\sigma) - \delta \\
&\geq -\left(\sigma^2 / 4 + \delta\right) + c_1(\sigma) - \delta \geq \delta.
\end{align*}

This completes the proof of the non-dissipative case (2). Thus the proof of Theorem 3.3 is complete. \( \square \)

Finally in this section, we consider the special case where \( g(u) \equiv 0 \) so that \( f(u) = \frac{1}{q} (-u)^{q+1} \). In this case the degenerate stationary wave is given explicitly by \( \phi(x) = -(1 + x)^{-1/q} \) and the operator \( L \) in (1.5) is reduced to \( L_0 \) below:
\[ L_0 v = v_{xx} + \frac{q + 1}{q} \left( \frac{v}{1 + x} \right)_x. \] (3.12)

For this simplest case, we have the complete characterization of the dissipativity of the operator \( L_0 \).
Theorem 3.5. Let $\alpha_c(q) := 3 + 2/q$. Then we have the complete characterization of the dissipativity of the operator $L_0$ given in (3.12):

1. Let $-1 < \alpha < \alpha_c(q)$. Then $L_0$ is uniformly dissipative in $L^2_{\alpha}$. Namely, there is a positive constant $\delta$ such that

$$\langle L_0v, v \rangle_{L^2_{\alpha}} \leq -\delta (\|v_x\|^2_{L^2_{\alpha}} + \|v\|^2_{L^2_{\alpha-2}})$$

(3.13)

for $v \in H^1_{\alpha,0}$.

2. Let $\alpha = \alpha_c(q)$ or $\alpha = -1$. Then $L_0$ is strictly dissipative in $L^2_{\alpha}$. Namely, we have $\langle L_0v, v \rangle_{L^2_{\alpha}} < 0$ for $v \in H^1_{\alpha,0}$ with $v \neq 0$.

3. Let $\alpha > \alpha_c(q)$ or $\alpha < -1$. Then $L_0$ can not be dissipative in $L^2_{\alpha}$. Namely, we have $\langle L_0v, v \rangle_{L^2_{\alpha}} > 0$ for some $v \in H^1_{\alpha,0}$ with $v \neq 0$.

Proof. Consider the case where $f(u) = \frac{1}{q} (-u)^q$ with $g(u) \equiv 0$. In this case we have $\phi(x) = -(1 + x)^{-1/q}$ and $L = L_0$. Moreover, noting that $r(u) \equiv 0$, we have as a counterpart of (3.7),

$$\langle L_0v, v \rangle_{L^2(w)} = -\|v_x\|^2_{L^2(w)} + c_1(\sigma) \|v\|^2_{L^2(w_1)};$$

(3.14)

where $w$ and $w_1$ are the weight functions defined in (2.9) with $\sigma \in \mathbb{R}$, and $c_1(\sigma)$ is given in (3.6). In our special case, these weight functions are given explicitly by (2.12), so that we have

$$\langle L_0v, v \rangle_{L^2(w)} = q \langle L_0v, v \rangle_{L^2_{\sigma/q+1}},$$

$$\|v_x\|^2_{L^2(w)} = q \|v_x\|^2_{L^2_{\sigma/q+1}}, \quad \|v\|^2_{L^2(w_1)} = \frac{1}{q} \|v\|^2_{L^2_{\sigma/q-1}}.$$  

(3.15)

Now we put $\sigma = (\alpha - 1)q$. First, let $-1 < \alpha < \alpha_c(q)$. This corresponds to the case where $-2q < \sigma < 2(q + 1)$, for which we have $c_1(\sigma) < \sigma^2/4$. Therefore, applying to (3.14) the same arguments as in (1) of Theorem 3.3, we obtain

$$\langle L_0v, v \rangle_{L^2(w)} \leq -\delta (\|v_x\|^2_{L^2(w)} + \|v\|^2_{L^2(w_1)})$$

for some $\delta > 0$. This inequality together with the relations in (3.15) (with $\sigma = (\alpha - 1)q$) shows the uniform dissipativity of $L_0$ in $L^2_{\alpha}$.

Second, let $\alpha = \alpha_c(q)$ or $\alpha = -1$. This corresponds to the case where $\sigma = 2(q + 1)$ or $\sigma = -2q$. In this case we have $c_1(\sigma) = \sigma^2/4$. On the other hand, we have $(\sigma^2/4)\|v\|^2_{L^2(w_1)} \leq \|v_x\|^2_{L^2(w)}$ by the Hardy type inequality (2.10). Consequently, we get from (3.14) that

$$\langle L_0v, v \rangle_{L^2(w)} \leq 0.$$
Here the equality holds if and only if \( (\sigma^2/4)\|v\|^2_{L^2(w)} = \|v_x\|^2_{L^2(w)} \). However, we know from Proposition 2.3 that such a \( v \neq 0 \) does not exist in \( H^1_0(w) \). Thus we conclude that \( \langle L_0 v, v \rangle_{L^2(w)} < 0 \) for \( v \in H^1_0(w) \) with \( v \neq 0 \), which together with (3.15) (with \( \sigma = (\alpha - 1)q \)) proves the strict dissipativity of \( L_0 \) in \( L^2_\alpha \).

Finally, let \( \alpha > \alpha_c(q) \) or \( \alpha < -1 \). Then we have \( \sigma > 2(q + 1) \) or \( \sigma < -2q \) and hence \( c_1(\sigma) > \sigma^2/4 \). Therefore, applying to (3.14) the same arguments as in (2) of Theorem 3.3, we find that \( \langle L_0 v, v \rangle_{L^2(w)} > 0 \) for some \( v \in H^1_0(w) \) with \( v \neq 0 \). This together with (3.15) (with \( \sigma = (\alpha - 1)q \)) gives the desired conclusion of (3) of Theorem 3.5. This completes the proof.

4 Nonlinear stability

The aim of this section is to prove the following stability result for the nonlinear problem (1.4) that is a refinement of the result in [15].

**Theorem 4.1.** Assume (1.2). Suppose that \( v_0 \in L^2_\alpha \cap L^\infty \) for some \( 1 \leq \alpha < \alpha_c(q) := 3 + q/2 \). Then there is a positive constant \( \delta_1 \) such that if \( \|v_0\|_{L^2_\alpha} \leq \delta_1 \), then the problem (1.4) has a unique global solution \( v \in C^0([0, \infty); L^2_\alpha \cap L^p) \) for each \( p \) with \( 2 \leq p < \infty \). Moreover, the solution verifies the decay estimate

\[
\|v(t)\|_{L^p} \leq C(\|v_0\|_{L^2_\alpha} + \|v_0\|_{L^\infty})(1 + t)^{-\alpha/4 - \nu} \tag{4.1}
\]

for \( t \geq 0 \), where \( 2 \leq p < \infty \), \( \nu = (1/2)(1/2 - 1/p) \), and \( C \) is a positive constant.

A key to the proof of Theorem 4.1 is to show the following space-time weighted energy inequality.

**Proposition 4.2.** Assume the same conditions as in Theorem 4.1. Let \( v \) be a solution to the problem (1.4) with the initial data \( v_0 \in L^2_\alpha \cap L^\infty \), where \( 1 \leq \alpha < \alpha_c(q) := 3 + 2/q \). Then there is a positive constant \( \delta_2 \) such that if \( \|v_0\|_{L^2_\beta} \leq \delta_2 \), then we have

\[
\|v(t)\|_{L^2_\beta} \leq C\|v_0\|_{L^2_\beta}. \tag{4.2}
\]

Moreover, we have the following space-time weighted energy inequality:

\[
(1 + t)^{\gamma}\|v(t)\|^2_{L^2_\beta} + \int_0^t (1 + \tau)^{\gamma}(\|v_x(\tau)\|^2_{L^2_\beta} + \|v(\tau)\|^2_{L^2_\beta})\,d\tau \\
\leq C\|v_0\|^2_{L^2_\beta} + \gamma C\int_0^t (1 + \tau)^{-1}\|v(\tau)\|^2_{L^2_\beta}\,d\tau \tag{4.3}
\]
for any $\gamma \geq 0$ and $\beta$ with $0 \leq \beta \leq \alpha$, where the constant $C$ is independent of $\gamma$ and $\beta$.

**Proof.** The main part of the proof of this proposition is to derive the following space-time weighted energy inequality:

\[(1 + t)^\gamma \|v(t)\|_{L^2_\beta}^2 + \int_0^t (1 + \tau)^\gamma (\|v_x(\tau)\|_{L^2_\beta}^2 + \|v(\tau)\|_{L^2_{\beta-2}}^2) \, d\tau \leq C\|v_0\|_{L^2_\beta}^2 + \gamma C \int_0^t (1 + \tau)^{\gamma-1}\|v(\tau)\|_{L^2_\beta}^2 \, d\tau + CS^\gamma_\beta(t) \]  

\[(4.4)\]

for any $\gamma \geq 0$ and $\beta$ with $0 \leq \beta \leq \alpha$, where $1 \leq \alpha < \alpha_c(q) := 3 + 2/q$, $C$ is a constant independent of $\gamma$ and $\beta$, and

\[S^\gamma_\beta(t) = \int_0^t (1 + \tau)^\gamma \|v(\tau)\|_{L^3_{\beta-1}}^3 \, d\tau. \]  

\[(4.5)\]

Once (4.4) is obtained, we can show the desired estimates (4.2) and (4.3) as follows. We observe that

\[S^\gamma_\beta(t) \leq C \int_0^t (1 + \tau)^\gamma \|v(\tau)\|_{L^2_\beta}^2 (\|v_x(\tau)\|_{L^2_\beta}^2 + \|v(\tau)\|_{L^2_{\beta-2}}^2) \, d\tau, \]  

\[(4.6)\]

which is an easy consequence of the following inequality (see [15] for the details):

\[\|v\|_{L^3_{\beta-1}}^3 \leq C\|v\|_{L^2_\beta}^2 \|v\|_{L^2_{\beta-2}} (\|v_x\|_{L^2_\beta}^2 + \|v\|_{L^2_{\beta-2}}^2), \]

where $\beta \in \mathbb{R}$. Now we put $\gamma = 0$ and $\beta = 1$ in (4.4) and define $V(t) \geq 0$ by

\[V(t)^2 = \sup_{0 \leq \tau \leq t} \|v(\tau)\|_{L^2_1}^2 + \int_0^t (\|v_x(\tau)\|_{L^1_1}^2 + \|v(\tau)\|_{L^2_{-1}}^2) \, d\tau. \]

Since $S^0_1(t) \leq CV(t)^3$ by (4.6), we get the inequality $V(t)^2 \leq C\|v_0\|_{L^2_1}^2 + CV(t)^3$. This can be solved as $V(t) \leq C\|v_0\|_{L^2_1}$, provided that $\|v_0\|_{L^2_1}$ is suitably small. Thus we obtain

\[\|v(t)\|_{L^2_1}^2 + \int_0^t (\|v_x(\tau)\|_{L^1_1}^2 + \|v(\tau)\|_{L^2_{-1}}^2) \, d\tau \leq C\|v_0\|_{L^2_1}^2, \]  

\[(4.7)\]

which gives the uniform estimate (4.2). Consequently, we have

\[S^\gamma_\beta(t) \leq C\|v_0\|_{L^2_1} \int_0^t (1 + \tau)^\gamma (\|v_x(\tau)\|_{L^2_\beta}^2 + \|v(\tau)\|_{L^2_{\beta-2}}^2) \, d\tau. \]
Substituting this estimate into (4.4) and assuming that $\|v_0\|_{L^2}^2$ is suitably small (say, $\|v_0\|_{L^2}^2 \leq \delta_2$), we arrive at the desired energy inequality (4.3).

It remains to prove the inequality (4.4). To this end, we recall the following uniform estimate:

$$\|v(t)\|_{L^\infty} \leq M_\infty,$$  \hspace{1cm} (4.8)

where $M_\infty = \|v_0\|_{L^\infty} + 2$. This is an easy consequence of the maximum principle (see [5] for the details).

**Proof of (4.4) for $\beta = 0$.** The proof is based on the time weighted $L^2$ energy method. We multiply the equation (1.4) by $v$. This yields

$$\left(\frac{1}{2}v^2\right)_t + (F - vv_x)_x + v_x^2 + G = 0,$$  \hspace{1cm} (4.9)

where

$$F = (f(\phi + v) - f(\phi))v - \int_0^v (f(\phi + \eta) - f(\phi))\,d\eta,$$

$$G = \int_0^v (f'(\phi + \eta) - f'(\phi))\,d\eta \cdot \phi_x.$$  \hspace{1cm} (4.10)

We note that

$$F = \frac{1}{2}f'(\phi)v^2 + O(|v|^3), \quad G = \frac{1}{2}f''(\phi)\phi_x v^2 + \phi_x O(|v|^3)$$  \hspace{1cm} (4.11)

for $v \to 0$. Also, we observe that

$$\phi_x = f(\phi) = \frac{1}{q}(-\phi)^{q+1}(1 + O(|\phi|)),$$

$$f''(\phi) = (q+1)(-\phi)^{q-1}(1 + O(|\phi|))$$

for $|\phi| \to 0$ and that $f''(\phi) > 0$ by (1.2). Therefore, noting (3.2) and (4.8), we have from (4.11) that

$$G \geq c(1 + x)^{-2}v^2 - C(1 + x)^{-1-1/q}|v|^3$$  \hspace{1cm} (4.12)

for any $x \in \mathbb{R}_+$. We integrate (4.9) over $\mathbb{R}_+$ and substitute (4.12) into the resulting equality. This gives

$$\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \|v_x\|_{L^2}^2 + c\|v\|_{L^2}^2 \leq C\|v\|_{L^2}^3.$$  

We multiply this inequality by $(1 + t)^\gamma$ and integrate with respect $t$. This yields the desired inequality (4.4) for $\beta = 0$. 

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Proof of (4.4) for $\beta > 0$. We apply the space-time weighted energy method employed in [15] (see also [3]). Let $w > 0$ be a smooth weight function depending only on $x$, which will be specified later. We multiply (4.9) by $w$, obtaining

$$
\left(\frac{1}{2} v^2 w\right)_t + \left\{ (F - \mu vv_x) w + \frac{1}{2} v^2 w_x \right\}_x + v_x^2 w - \left(\frac{1}{2} v^2 w_{xx} + Fw_x - Gw\right) = 0.
$$

(4.13)

Here, using (4.11), we have

$$
\frac{1}{2} v^2 w_{xx} + Fw_x - Gw = \frac{1}{2} v^2 (w_{xx} + w_x f'(\phi) - w f''(\phi) \phi_x) + R,
$$

(4.14)

where $R = w_x O(|v|^3) - w \phi_x O(|v|^3)$ for $v \to 0$. Notice that the coefficient $w_{xx} + w_x f'(\phi) - w f''(\phi) \phi_x$ in (4.14) is just the same as that appeared in (3.4).

Now we choose the weight function $w$ and the corresponding $w_1$ by (2.9) with $\sigma = (\beta - 1)q$, where $0 \leq \beta \leq \alpha$ and $1 \leq \alpha < \alpha_c(q) := 3 + 2/q$. Then we have (3.5) with $\sigma = (\beta - 1)q$. Substituting these expressions into (4.13) and integrating over $\mathbb{R}_+$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \|v\|_{L^2(w)}^2 + \|v_x\|_{L^2(w)}^2 - c_1(\sigma) \|v\|_{L^2(w_1)}^2 + \int_0^\infty v^2 r(\phi) w_1 dx = \int_0^\infty R dx,
$$

(4.15)

where $c_1(\sigma)$ and $r(\phi)$ are given in (3.6) with $\sigma = (\beta - 1)q$. Here our weight functions in (2.9) are

$$
w = q(-\phi)^{-\sigma-q}/(1 + g(\phi)), \quad w_1 = \frac{1}{q}(-\phi)^{-\sigma+q}(1 + g(\phi)).
$$

Therefore, noting (3.2), we see that

$$
w \sim (1 + x)^{\sigma/q+1} = (1 + x)^{\beta}, \quad w_1 \sim (1 + x)^{\sigma/q-1} = (1 + x)^{\beta-2},
$$

(4.16)

where the symbol $\sim$ means the equivalence. This implies that the norms $\| \cdot \|_{L^2(w)}$ and $\| \cdot \|_{L^2(w_1)}$ are equivalent to $\| \cdot \|_{L^2_\beta}$ and $\| \cdot \|_{L^2_{\beta-2}}$, respectively.

We estimate (4.15) similarly as in (1) of Theorem 3.3. To this end, we note that $\sigma_1 \leq \sigma \leq \sigma_2$, where $\sigma_1 = -q$ and $\sigma_2 = (\alpha - 1)q$. Since $c_1(\sigma) < \sigma^2/4$ for $-2q < \sigma < 2(q + 1)$ and since $-2q < \sigma_1 < \sigma_2 < 2(q + 1)$, we can choose $\delta > 0$ so small that

$$
\delta \leq \min_{\sigma_1 \leq \sigma \leq \sigma_2} \frac{\sigma^2/4 - c_1(\sigma)}{2 + \sigma^2/4}.
$$
Notice that this $\delta$ is independent of $\beta$. For this choice of $\delta$, we take $a = a(\delta) > 0$ so large that $|r(\phi)| \leq \delta$ for $x \geq a$. Then we have

$$\left| \int_0^\infty v^2 r(\phi) w_1 \, dx \right| \leq \delta \|v\|_{L^2(w_1)}^2 + C \|v\|_{L^2(w)}^2,$$

where $C$ is a constant satisfying $C \geq (1 + x)^2 |r(\phi)| w_1$ for $0 \leq x \leq a$. Also, using the Hardy type inequality $(\sigma^2/4) \|v\|_{L^2(w_1)}^2 \leq \|v_x\|_{L^2(w)}^2$ in (2.10), we have

$$\|v_x\|_{L^2(w)}^2 - c_1(\sigma) \|v\|_{L^2(w_1)}^2 = \delta \|v_x\|_{L^2(w)}^2 + (1 - \delta) \|v_x\|_{L^2(w)}^2 - c_1(\sigma) \|v\|_{L^2(w_1)}^2$$

$$\geq \delta \|v_x\|_{L^2(w)}^2 + \{(1 - \delta)\sigma^2/4 - c_1(\sigma)\} \|v\|_{L^2(w_1)}^2$$

$$\geq \delta \|v_x\|_{L^2(w)}^2 + 2\delta \|v\|_{L^2(w_1)}^2,$$

where we used the fact that $(1 - \delta)\sigma^2/4 - c_1(\sigma) \geq 2\delta$. On the other hand, using (4.8), we see that $|R| \leq C(\|w_x\| + w\phi_x) |v|^3$. Moreover, a straightforward computation shows that $|w_x| + w\phi_x \leq C(1 + x)^{\beta - 1}$. Substituting all these estimates into (4.15), we obtain

$$\frac{1}{2} \frac{d}{dt} \|v\|_{L^2(w)}^2 + \delta(\|v_x\|_{L^2(w)}^2 + \|v\|_{L^2(w_1)}^2) \leq C \|v\|_{L^2_2}^2 + C \|v\|_{L^2_{\beta - 1}}^3,$$  

where $\delta$ and $C$ are independent of $\beta$. We multiply this inequality by $(1 + t)^\gamma$ and integrate with respect to $t$. By virtue of (4.16), we have

$$(1 + t)^\gamma \|v(t)\|_{L^2_\beta}^2 + \int_0^t (1 + \tau)^\gamma (\|v_x(\tau)\|_{L^2_\beta}^2 + \|v(\tau)\|_{L^2_{\beta - 2}}^2) \, d\tau$$

$$\leq C \|v_0\|_{L^2_\beta}^2 + \gamma C \int_0^t (1 + \tau)^{\gamma - 1} \|v(\tau)\|_{L^2_\beta}^2 \, d\tau$$

$$+ C \int_0^t (1 + \tau)^\gamma \|v(\tau)\|_{L^2_2}^2 \, d\tau + CS_\beta^\gamma(t),$$

where the constant $C$ is independent of $\gamma$ and $\beta$. Here the third term on the right hand side of (4.18) was already estimated by (4.4) with $\beta = 0$. Hence we have proved (4.4) also for $0 < \beta \leq \alpha$. This completes the proof of Proposition 4.2.

**Proof of Theorem 4.1.** The proof is essentially the same as that of Theorem 3.3 of [15] so that we only give an outline of the proof of the decay estimate (4.1).
First, we show the following $L^2$ decay estimate by applying the induction argument to the space-time weighted energy inequality (4.3).

$$(1 + t)^j \| v(t) \|^2_{L^2_{\alpha - 2j}} + \int_0^t (1 + \tau)^j (\| v_x(\tau) \|^2_{L^2_{\alpha - 2j}} + \| v(\tau) \|^2_{L^2_{\alpha - 2j - 2}}) \, d\tau \leq C \| v_0 \|^2_{L^2_{\alpha}}$$

(4.19)

for each integer $j$ with $0 \leq j \leq \lfloor \alpha/2 \rfloor$. To see this, we put $\gamma = j$ and $\beta = \alpha - 2j$ in (4.3), obtaining

$$(1 + t)^j \| v(t) \|^2_{L^2_{\alpha - 2j}} + \int_0^t (1 + \tau)^j (\| v_x(\tau) \|^2_{L^2_{\alpha - 2j}} + \| v(\tau) \|^2_{L^2_{\alpha - 2j - 2}}) \, d\tau \leq C \| v_0 \|^2_{L^2_{\alpha}} + jC \int_0^t (1 + \tau)^{j - 1} \| v(\tau) \|^2_{L^2_{\alpha - 2j}} \, d\tau.$$  

(4.20)

Then, applying the induction with respect to the integer $j$ with $0 \leq j \leq \lfloor \alpha/2 \rfloor$, we obtain (4.19).

On the other hand, when $\alpha/2$ is not an integer, we have

$$(1 + t)^\gamma \| v(t) \|^2_{L^2} + \int_0^t (1 + \tau)^\gamma (\| v_x(\tau) \|^2_{L^2} + \| v(\tau) \|^2_{L^2_{\alpha - 2j - 2}}) \, d\tau \leq C \| v_0 \|^2_{L^2} + jC \int_0^t (1 + \tau)^{j - 1} \| v(\tau) \|^2_{L^2_{\alpha - 2j}} \, d\tau.$$  

(4.21)

for any $\gamma$ with $\gamma > \alpha/2$. This can be shown by using (4.3) with $\gamma > \alpha/2$, $\beta = 0$ and (4.19) with $j = \lfloor \alpha/2 \rfloor$ together with Nishikawa’s technique in [12]. For the details, see the proof of Proposition 2.6 of [15].

Thus we have shown the following $L^2$ decay estimate:

$$\| v(t) \|_{L^{2}} \leq C \| v_0 \|_{L^2_{\alpha}} (1 + t)^{\eta/4}. \quad (4.22)$$

The desired $L^p$ decay estimate (4.1) is then obtained by the time weighted $L^p$ energy method. In fact, under the additional smallness condition on $\| v \|_{L^2_{\alpha}}$, we have

$$(1 + t)^\gamma \| v(t) \|^p_{L^p} + \int_0^t (1 + \tau)^\gamma (\| (|v|^{p/2})_x(\tau) \|^2_{L^2} + \| v(\tau) \|^p_{L^p_{\alpha - 2j - 2}}) \, d\tau \leq C \| v_0 \|^p_{L^p} + C \| v_0 \|^p_{L^2_{\alpha}} (1 + t)^{\eta/4} \| v \|^p_{L^p_{\alpha - 2j}}.$$  

(4.23)

where $2 \leq p < \infty$, $\nu = (1/2)(1/2 - 1/p)$ and $\gamma > (\alpha/4 + \nu)p$. We omit the details and refer the reader to the proof of Theorem 2.3 of [15]. This completes the proof of Theorem 4.1.
References


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