Analysis of the HUM Control Operator
Exact Controllability for the Semilinear Waves

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Exact controllability for the waves

\[ (W) \begin{cases} \partial_t^2 u - \Delta u = 0 \quad \text{in} \quad ]0, +\infty[ \times M \\ (u(0), \partial_t u(0)) = U_0 = (u_0, u_1) \in H^1 \times L^2 \end{cases} \]
Exact controllability for the waves

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(u(0), \partial_t u(0)) &= U_0 = (u_0, u_1) \in H^1 \times L^2
\end{aligned}
\]

- \( M \) Riemannian manifold, connected, compact, without boundary, with dimension \( d \)
  (ex: sphere, torus...)

Dehman-Lebeau ()

HUM analysis
Exact controllability for the waves

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- \( M = \Omega \) open subset of \( \mathbb{R}^d \), connected, bounded and smooth
  (homogeneous Dirichlet condition on the boundary)
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$$E = C([0, +\infty[, H^1) \cap C^1([0, +\infty[, L^2)$$
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E = C([0, +\infty[, H^1) \cap C^1([0, +\infty[, L^2)
\]

\( \omega \) open subset of \( M \) and \( T > 0 \) (suitable)
The Goal

\[(v_0, v_1) \in E = H^1 \times L^2\]

Find a source \(f(t, x) \in L^1_{loc}([0, +\infty[, L^2), \text{supp} f \subset \omega\)

\[
\begin{cases}
\partial_t^2 u - \Delta u = f \\
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\[A : (u_0, u_1) \rightarrow f\]
The HUM method

\[ f = \chi(x) g \approx 1_{\omega} g \]
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\[
\begin{aligned}
\partial_t^2 g - \Delta g &= 0 \\
G(T) &= G_0 = (g_0, g_1) \in E_{-1} = L^2 \times H^{-1}
\end{aligned}
\]

And we solve

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\partial_t^2 u - \Delta u &= \chi(x)g \\
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\[ \begin{cases} G_0 \to (u(T), \partial_t u(T)) \end{cases} \]

\[ \int_M (\overline{g_0} \partial_t u(T) - \overline{g_1} u(T)) dx = \int_0^T \int_M \chi(x) |g|^2 \, dx \, dt \]

\[ \langle G_0, SG_0 \rangle_{E_{-1}, E} = \int_0^T \int_M \chi(x) |g|^2 \, dx \, dt \]
Observation

\[ \int_0^T \int_M \chi(x) \vert g \vert^2 \, dx \, dt \geq C \| G_0 \|_{E^{-1}}^2 \]

\[ \| S G_0 \|_E \geq C \| G_0 \|_{E^{-1}} \quad \text{and} \quad S : E^{-1} \leftrightarrow E \]
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Conclusion: For \( U_0 \in E = H^1 \times L^2 \), we find a control \( f = \chi(x) g \in L^1([0, T], L^2) \).
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→ The controlled solution \( U(t) = (u, \partial_t u) \in E = H^1 \times L^2 \)
Observation
\[
\int_0^T \int_M \chi(x) |g|^2 \, dx \, dt \geq C \| G_0 \|^2_{E^{-1}} \\
\| SG_0 \|_E \geq C \| G_0 \|_{E^{-1}} \quad \text{and} \quad S : E^{-1} \leftrightarrow E
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Remark: This HUM control is solution of a variational problem: it’s the only one that minimizes
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\int_0^T \int_\omega |f|^2 \, dx \, dt
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**Definition:** We will call HUM control operator the operator \( \Lambda \) defined from \( E \) into \( E_{-1} \) by

\[ G_0 = \Lambda U_0 \]
Two problems

a) Control of smooth data

\[ U_0 \in E_k = H^{k+1} \times H^k, \quad k \geq 0 \]
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*Bardos-Lebeau-Rauch (92)*: Observation in each \( H^s(M) \).

b) Treatment of the frequencies

→ Does the control \( \Lambda U_0 \) load the frequencies carrying the data?
→ If \( U_0 \) has only low frequencies, how are the high frequencies of \( \Lambda U_0 \)?
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Control of nonlinear equations

\[ \partial_t^2 u - \Delta u + f(u) = 0 \]

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Further Question

Control of nonlinear equations

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\[ \rightarrow \] \( f \) lipschitz: control to zero, in uniform time, of small datas (Zuazua 92').

\[ \text{Dehman-Lebeau} \]
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→ \( f \) lipschitz : control to zero, in uniform time, of small datas (Zuazua 92').
→ \( f \) subcritical: stabilization → small datas....: non uniform control time (Dehman-Lebeau-Zuazua 03') et (Dehman-Gérard-Lebeau 06' for Schrödinger ).
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Program
→ Sharp analysis of the control operator $\Lambda : E \rightarrow E_{-1}$
→ Quantification of $\Lambda$. 
An abstract control

\[ \exp(itA), \ t \geq 0, \ \text{a semi-group of contractions on a hilbert space } H. \]
\[ B \ \text{bounded operator on } H \ \text{and } g \in L^1([0, T], H) \]

\[
(\partial_t - iA)f = Bg, \quad f(0) = 0
\]

\[
f(T) = \int_0^T e^{i(T-t)A}Bg(t)dt
\]

We choose \( g \) solution of

\[
(\partial_t - iA^*)g = 0, \quad g(T) = g_0
\]

**Observation**

\[
\int_0^T \left\| B^* e^{-itA^*} h \right\|_H^2 dt \geq C \left\| h \right\|_H^2
\]

The HUM optimal control is given by

\[
g(t) = B^* e^{-i(T-t)A^*} g_0
\]
\[ f_0 = f(T) = M_T g_0 \]

\[ M_T = \int_0^T e^{i(T-t)A} BB^* e^{-i(T-t)A^*} \, dt = \int_0^T e^{itA} BB^* e^{-itA^*} \, dt \]

If \( A \) is self-adjoint

\[ M_T = \int_0^T e^{itA} BB^* e^{-itA} \, dt = \Lambda^{-1} \]

\[ m(t) = e^{itA} BB^* e^{-itA} \]
Notations

\( M = \Omega \) open bounded, connected and smooth subset of \( \mathbb{R}^d \).

\((e_j, \omega_j^2)_{j \geq 1}\) the spectral elements of \( \Omega \).

\[-\Delta e_j = \omega_j^2 e_j, \quad e_j|_{\partial \Omega} = 0, \quad \|e_j\|_{L^2(\Omega)} = 1\]

\[H^s(\Delta) = \{u = \sum_j a_j e_j, \quad \sum_j (1 + \omega_j^2)^s |a_j|^2 < \infty\}\]

\[\lambda = \lambda(x, D_x) = \sqrt{|\Delta|}\]

\[\lambda(x, D_x) \sum_j a_j e_j = \sum_j \omega_j a_j e_j\]

If \( M \) is a compact manifold without boundary, it is a pseudo-differential operator of order 1 (Helffer-Sjöstrand formula).
Let $\varphi \in C_0^\infty(\mathbb{R})$, and $\psi \in C_0^\infty(\mathbb{R}^*)$ such that

$$\varphi(s) + \sum_{k=1}^{\infty} \psi(2^{-k}s) = 1, \quad s \in \mathbb{R}$$

$$\begin{cases} 
\psi_0(s) = \varphi(s) \\
\psi_k(s) = \psi(2^{-k}s), \quad k \geq 1 
\end{cases}$$
Littlewood-Paley Decomposition

Let \( \varphi \in C_0^\infty(\mathbb{R}) \), and \( \psi \in C_0^\infty(\mathbb{R}^*) \) such that

\[
\varphi(s) + \sum_{k=1}^{\infty} \psi(2^{-k}s) = 1, \quad s \in \mathbb{R}
\]

\[
\begin{align*}
\psi_0(s) &= \varphi(s) \\
\psi_k(s) &= \psi(2^{-k}s), \quad k \geq 1
\end{align*}
\]

**Spectral localization operators**

\( k \in \mathbb{N} \), \( u = \sum_j a_j e_j \),

\[
\psi_k(D)u = \sum_j \psi_k(\omega_j)a_j e_j = \sum_j \psi(2^{-k}\omega_j)a_j e_j
\]

\[
S_k(D) = \sum_{j=0}^{k} \psi_j(D) = \psi_0(2^{-k}D), \quad k \geq 0
\]

→ If \( M \) is compact, these are pdo of order 0
→ They commute to the laplacian.
Time dependant control

\[
\begin{cases}
\partial^2_t u - \Delta u = \chi g \\
U_0 = (0, 0)
\end{cases}
\]

\[
\chi(t, x) = \varphi(t)\chi_0(x) \text{ in } C^\infty(\mathbb{R} \times \overline{\Omega}),
\]

\[
\varphi \text{ flat at } t = 0, T \text{ and } \varphi > 0 \text{ in } ]0, T[.
\]

\[
\omega = \{ x \in \Omega, \chi_0(x) \neq 0 \}
\]
Time dependant control

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\]

**Geometric Control Condition (GCC)**

\(\omega\) satisfies (GCC) at time \(T\) if every geodesic ray of \(\Omega\) travelling with speed 1 and starting at \(t = 0\), enters the open set \(\omega\) in a time \(t < T\).
\[ E = H^1_0 \times L^2 \leftrightarrow L^2 \otimes L^2 \]

\[ u_0 = \lambda^{-1}(h_+ + h_-), \quad u_1 = i(h_+ - h_-) \]

\[ \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \quad \begin{pmatrix} \chi & \chi \\ -\chi & -\chi \end{pmatrix} \]

\[(\partial_t - iA)f = Bg, \quad f(0) = 0\]

\[ M_T = \frac{1}{2} \int_0^T \begin{pmatrix} e^{it\lambda} \chi^2 e^{-it\lambda} & -e^{it\lambda} \chi^2 e^{it\lambda} \\ -e^{-it\lambda} \chi^2 e^{-it\lambda} & e^{-it\lambda} \chi^2 e^{it\lambda} \end{pmatrix} dt = \frac{1}{2} \begin{pmatrix} Q_+ & T_+ \\ T_- & Q_- \end{pmatrix} \]

\[ Q_\pm = \int_0^T e^{\pm it\lambda} \chi^2 e^{\mp it\lambda} dt, \quad T_\pm = -\int_0^T e^{\pm it\lambda} \chi^2 e^{\pm it\lambda} dt \]
Statement of the results

**Theorem 1: Quantification of $\Lambda$**

Under (GCC), $Q_{\pm}$ is an isomorphism on $H^s(\Delta)$ for every $s \geq 0$ and

$$T_\pm : H^s(\Delta) \to H^{s+1}(\Delta)$$

Moreover, with $L_{\pm} = Q_{\pm}^{-1}$

$$\Lambda = \begin{pmatrix} 2L_+ & 0 \\ 0 & 2L_- \end{pmatrix} + R$$

where $R$ is smoothing.

In particular $\Lambda$ is an isomorphism on $H^s(\Delta) \otimes H^s(\Delta)$. 

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**Theorem 2:** Under (GCC),

$$\|\psi_k(D)\Lambda - \Lambda\psi_k(D)\|_{L^2} \leq c2^{-k}$$

$$\|S_k(D)\Lambda - \Lambda S_k(D)\|_{L^2} \leq c2^{-k}$$
Remarks

1. Finally, the HUM control $\Lambda$ preserves automatically the regularity of the data to be controlled.

2. $\Lambda$ acts almost individually on each frequency block of the solution.

3. $\Lambda$ has a pseudo-differential behavior.

4. $(e_n, e_n) \rightarrow (e_{n+1}, e_{n+1})$: the control vector and the controlled solution live at frequency $\omega_n$, for $n$ large enough.

5. Numerical applications: G.Lebeau-M.Nodet (08’).
Case of a compact manifold without boundary

$(e_j, \omega_j^2)_{j \geq 0}$ the spectral elements of $M$.

\[ \Pi_+ \sum_{j \geq 0} a_j e_j = \sum_{j \geq 1} a_j e_j \]

For $(x, \xi) \in T^* M \setminus 0$, we denote

\[ \gamma(s) = \gamma_{(x, \xi)}(s), \quad s \in \mathbb{R} \]

the bicharacteristic of the wave operator starting from $(x, \xi)$.

\[ \alpha(x, \xi) = \left( \int_0^T \chi^2(\gamma(s)) ds \right)^{-1} \]

\[ \beta(x, \xi) = \alpha(x, -\xi) \]

Under (GCC), these are elliptic pseudo-differential symbols of order 0.
Theorem 3
Under (GCC), $\Lambda$ is an elliptic pseudo-differential operator of order 0,

$$
\Lambda = \Pi_+ \begin{pmatrix} 2\alpha(x, D) & 0 \\ 0 & 2\beta(x, D) \end{pmatrix} \Pi_+ + R
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where $R$ is a smoothing pdo.

$\rightarrow$ $\Lambda$ is an isomorphism on $H^s(M) \otimes H^s(M)$.

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**Theorem 3**

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**Corollary** : With the same assumptions, if $U_0 = (u_0, u_1)$ and $G_0 = (g_0, g_1) = \Lambda U_0$, $s \geq 0$,

$$WF_{s+1}(u_0) \cup WF_s(u_1) = WF_s(g_0) \cup WF_{s-1}(g_1)$$

$\rightarrow$ $\Lambda$ is microlocal.

$\rightarrow$ Analogous result on a domain $\Omega$ of $\mathbb{R}^d$ by G.Lebeau-J.Rauch (work in progress).
Semilinear wave equation

\( M = \Omega \) open, bounded and smooth subset of \( \mathbb{R}^3 \), \( 1 \leq p < 5 \)

\( f : \mathbb{R} \to \mathbb{R} \), of class \( C^3 \), \( f(0) = 0 \), \( sf(s) \geq 0 \) \( \forall s \in \mathbb{R} \)

\[ |f^{(j)}(s)| \leq C(1 + |s|)^{p-j}, \quad \text{for} \quad j = 1, 2, 3 \]

**Theorem 4: Semilinear waves**

Under condition (GCC), for every \( C_0 > 0 \), there exist \( r > 0 \) and \( k \in \mathbb{N} \), s.t for every \( U_0, U_1 \) satisfying

\[ \| U_0 \|_{H^1_0 \times L^2} \leq C_0, \quad \| U_1 \|_{H^1_0 \times L^2} \leq C_0 \]

\[ \| S_k(D) U_0 \|_{H^1_0 \times L^2} \leq r, \quad \| S_k(D) U_1 \|_{H^1_0 \times L^2} \leq r \]

there exists \( g \in L^1(0, T; L^2) \) which exactly controls the semilinear wave equation at time \( T \), namely, the unique solution of system

\[
\begin{cases}
\Box u + f(u) = \chi g \quad \text{sur } \Omega, \quad u|_{\partial \Omega} = 0 \\
(u(0), \partial_t u(0)) = U_0
\end{cases}
\]

satisfies \( U(T) = U_1 \).
Remarks
1- The low frequencies determine the nonlinear behavior of the equation.
2- For high frequencies, we have a "linear behavior".
3- The control process is achieved in uniform time: the one of linear control.
4- Condition on LF energy ? Open problem.
5- Method of proof: Fixed point process used near the linear control.
6- Main tools: Strichartz estimates and analysis of LF of the linear control.
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Key lemma : Regularity of the nonlinear term
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Key lemma : Regularity of the nonlinear term

Corollary: In the same framework, if \((U_0^n)\) and \((U_1^n)\) are two sequences of data weakly converging to zero in \(H_0^1 \times L^2\), one can drive one of them to the other, by the semilinear wave evolution, at geometric control time. In particular, one can drive \((U_0^n)\) to 0.
Proof of Theorem 1

\[
(\Lambda^{-1}) = M_T = \frac{1}{2} \begin{pmatrix} Q_+ & T_+ \\ T_- & Q_- \end{pmatrix}
\]

\[
Q_+ = \int_0^T e^{it\lambda} \chi^2 e^{-it\lambda} dt, \quad T_+ = -\int_0^T e^{it\lambda} \chi^2 e^{it\lambda} dt
\]

**Difficulty**: \(\chi^2\) does not operate on the Sobolev spaces \(H^s(\Delta)\)

1. \(Q_+\) is an isomorphism on \(L^2\) : (GC) + Observation estimate
2. \(Q_+\) is bounded on \(H^s(\Delta)\)

\[
Q_+ = \sum_{i,j} C_{ij}, \quad C_{ij} = \int_0^T e^{it\lambda} \psi_i(D) \chi^2 \psi_j(D) e^{-it\lambda} dt
\]

The matrix \((2^{is} \| C_{ij} \|_{L^2 \rightarrow L^2} 2^{-js})\) is bounded on \(l^2(\mathbb{N})\).
\[ C_{ij} = \int_0^T e^{it\lambda} \lambda^{-1} \psi_i(D)(i\partial_t \chi^2 + \chi^2 \lambda)\psi_j(D)e^{-it\lambda} dt \]

\[ 2^{is} \| C_{ij} \|_{L^2 \to L^2} 2^{-js} \leq C_N 2^{-(N-s)|i-j|}, \quad N \geq 1 \]

3. \( Q_+ \) is an isomorphism on \( H^s(\Delta) \)

\[ u \in L^2 \text{ and } Q_+ u \in H^s(\Delta) \Rightarrow u \in H^s(\Delta) \]

\[ F_s = \{ u \in L^2 \quad \text{s.t.} \quad Q_+ u \in H^s(\Delta) \} \]

a) \( H^s(\Delta) \) is closed in \( F_s \)

\[ \| u \|_{H^s} \leq c(\| Q_+ u \|_{H^s} + \| u \|_{L^2}) \quad \forall u \in H^s(\Delta) \]

b) \( H^s(\Delta) \) is dense in \( F_s \) (regularization).
Lemma 1: For $a(x) \in C^\infty(\overline{\Omega})$,

$$\left\| [\psi_j(D), a(x)] \right\|_{L^2 \rightarrow L^2} \leq c2^{-j}$$

Lemma 2:

i) The bracket $[Q_+, \lambda^{-s}]$ is bounded from $L^2$ to $H^{s+1}(\Delta)$ for $s \in [0, 2[$.

ii) The bracket $[Q_+, \lambda]$ is bounded on $L^2$.

iii) For every $s \geq 0$, the operator

$$A_s = \lambda^s Q_+ \lambda^{-s} - Q_+$$

is bounded from $L^2$ to $H^{1/2}(\Delta)$.

In particular, $A_s$ is compact on $L^2$. 
Proof of nonlinear control

\[ f(u) = f'(0)u + \theta(u) \]

\[
\begin{cases}
\Box u + f(u) = \chi^2(g_1 + g_2) \\
U_0 \in H^1_0 \times L^2
\end{cases}
\]

\[ \Box g_1 + f'(0)g_1 = 0 \text{ and } G_1(0) = \Lambda(U_0, U_1) \]

\[ u = v + w \]

\[ \Box v + f'(0)v = \chi^2 g_1 \text{ and } V(0) = U_0 \]

then \( V(T) = U_1 \)

\[ \Box w + f'(0)w = -\theta(u) + \chi^2 g_2 \text{ and } W(0) = 0 \]

Dehman-Lebeau ()

HUM analysis
Let $h$ be solution of

$$\Box h + f'(0) h = \theta(u) \quad \text{and} \quad H(T) = 0$$

The function $k = w + h$ is solution of

$$\Box k + f'(0) k = \chi^2 g_2 \quad \text{and} \quad K(0) = H(0)$$

→ Now, the goal is to control this linear system

$$H(0) \to 0$$

∃? $G_2 \in L^2 \times H^{-1}$ such that

$$AG_2 = H(0) = K(0) = \Lambda^{-1} G_2$$

→ Fixed point in $L^2 \times H^{-1}$ for $L = \Lambda A$

→ $L$ is compact and reproduces a small ball $B_\rho$ centred at the origin of $L^2 \times H^{-1}$. 

Egorov Theorem

\[ \begin{cases} \partial_t u = iA(x; D_x)u & \text{in } \mathbb{R} \times M \\ u(0) = u_0 \end{cases} \]

\[ A = A_1 + A_0, \quad A_1(x; \xi) \in S^1_{cl} \text{ real, } A_0(x; \xi) \in S^0_{cl} \]

\[ A_1(x; \xi) \text{ homogeneous in } \xi \text{ for } |\xi| \geq 1 \]

\[ u(t, x) = U(t)u_0 \]

\( U(t) \) is bounded on each \( H^\sigma(M) \), with inverse \( U(-t) \).

**Egorov Theorem**

If \( P_0 = p_0(x, D) \in OPS^m_{1,0} \), then for every \( t \), the operator

\[ P(t) = U(t)P_0U(-t) \]

belongs to \( OPS^m_{1,0} \), modulo a smoothing operator. The principal symbol of \( P(t) \) (mod \( S^{m-1}_{1,0} \)) at \((x_0, \xi_0)\) is equal to \( p_0(\gamma(t)) \) where \( \gamma \) is the bicharacteristic of \( A_1 \) issued from \((x_0, \xi_0)\).
Strichartz Inequalities

**Theorem (Burq-Lebeau-Planchon 07’):** For every $T > 0$ and $E_0 > 0$, the system

\[
\begin{cases}
\Box u + f(u) = g & \text{in } ]0, +\infty[ \times \Omega \\
u = 0 & \text{on } ]0, +\infty[ \times \partial \Omega \\
\| u(0) \|_{H^1_0} + \| \partial_t u(0) \|_{L^2} \leq E_0
\end{cases}
\]

admits a unique solution $u$ in the class $C^0(0, T; H^1_0(\Omega)) \cap C^1(0, T; L^2(\Omega))$ satisfying

\[
\| u \|_{L^5(0, T; W^{3/5, 5})} \leq C
\]

for some $C = C(T, E_0, \| g \|_{L^1(0, T; L^2)})$.

In particular, for every $q \in [5, +\infty]$, $\exists C’ > 0$,

\[
\| u \|_{L^q(0, T; L^{3r})} \leq C'
\]

with $1/q + 1/r = 1/2$. 
Examples: \( L^{\infty}(L^{6}) \), \( L^{5}(L^{10}) \), \( L^{8}(L^{8}) \), ...

**Composition theorem**

If \( \nu(t,x) \in L^{5}(0, T; W^{\frac{3}{10}, 5}_{10}(\Omega)) \) then \( f(\nu) \in L^{1}(0, T; H^{\mu}(\Omega)) \), with \( \mu = \frac{3}{10}(5 - p) \).