Long time dynamics near the symmetry breaking bifurcation for Nonlinear Schrödinger/Gross-Pitaevskii Equations

Joint work with Michael Weinstein (Columbia University)

Jeremy Marzuola

Mathematical Institute
Universität Bonn

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Outline

Preliminaries

A coupled finite-infinite dimensional system

Finite Dimensional Hamiltonian Truncation

Conclusion
The Problem

\[
\begin{aligned}
  i\partial_t u &= (-\Delta + V(x))u + g(x)K[u\bar{u}]u, \\
  u(x, 0) &= u_0(x),
\end{aligned}
\]

Specifically, we select $V$ such that the discrete spectrum of the operator $H = -\Delta + V$ is well-understood.

This equation models the behavior or optical pulses in a waveguide or Bose-Einstein Condensates (BEC) in magnetic trap. In the particular case of the double well potential, physicists have observed high speed oscillations of optical pulses between the two wells.
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This equation models the behavior or optical pulses in a waveguide or Bose-Einstein Condensates (BEC) in magnetic trap. In the particular case of the double well potential, physicists have observed high speed oscillations of optical pulses between the two wells.
There exist $\Omega_0 < \Omega_1 < 0$ and functions $\psi_0, \psi_1$ such that

$$H\psi_j = \Omega_j \psi_j, \ j = 1, 2$$

where $\psi_0$ is symmetric about the origin and $\psi_1$ is anti-symmetric.

Explicitly, $V$ may be of the form

$$V(x) = \left[ \frac{1}{\sqrt{4\pi\sigma^2}} e^{-\frac{(x-L)^2}{4\sigma^2}} + \frac{1}{\sqrt{4\pi\sigma^2}} e^{-\frac{(x+L)^2}{4\sigma^2}} \right],$$

for some $L, \sigma$.

From henceforward, we define the following projections

$$P_0 f = \langle f, \psi_0 \rangle \psi_0,$$

$$P_1 f = \langle f, \psi_1 \rangle \psi_1,$$

$$P_c f = (I - P_0 - P_1)f.$$
Spectral Analysis

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Following the work of E. Harrell (1980), let us one start with a single rapidly decaying potential well centered at 0, say $V_0$, such that $H_0 = -\Delta + V_0$ has exactly one eigenvalue $\omega$.

Then, there exists $L_0$ such that for $L > L_0$, $H_L$ has a pair of simple eigenvalues, $\Omega_0 = \Omega_0(L)$ and $\Omega_1 = \Omega_1(L)$ and corresponding eigenfunctions $\psi_0$ (even) and $\psi_1$ (odd).

Moreover, for $L$ sufficiently large $|\Omega_0 - \Omega_1| = O(e^{-\kappa L})$, $\kappa > 0$.

This can in fact be generalized. Namely, if $H_0$ has $m$ bound states, $H_L$ will have $m$ pairs of eigenfunctions ($\Omega_{2j}$, $\Omega_{2j+1}$ with $\psi_{2j}$ symmetric and $\psi_{2j+1}$ anti-symmetric eigenfunctions for $j = 0, 1, \ldots m - 1$.

Similar results can be seen even if $V_0 = \delta_0(x)$. 

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Symmetry Breaking

Using the results from Kirr-Kevrekidis-Shlizerman-Weinstein (2008), we are able to describe steady state solutions of the form

$$\Psi_\Omega = (c_0 \psi_0 + c_1 \psi_1 + R(t))e^{-i\Omega t},$$

where $\psi_0, \psi_1, \Omega_0, \Omega_1$ are defined above and $|c_0|^2 + |c_1|^2 + \int |R|^2(t)dx = N$.

It is known that there exists $N_{\text{crit}}$ such that for $N < N_{\text{crit}}$, $\psi_0$ is stable under small generic perturbations, while for $N > N_{\text{crit}}$, the asymmetric state consisting of contributions from both $\psi_0$ and $\psi_1$ is stable.

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Symmetry Breaking Cont.

Figure: A numerical plot of the symmetry breaking in the soliton curve. Specifically, we plot here \( N \) versus \( \Omega \) for a double Gaussian potential with \( L = 3 \) and \( \sigma = 1 \).
Result Outline

- In this note we study the existence and long time stability of solutions consisting of fast oscillations between the wells.
- First of all, we develop the finite dimensional dynamical system to describe the oscillations.
- Then, we look at how such a finite dimensional solution compares to the full solution to the nonlinear PDE.
- The methods contained here apply to solutions which are in some sense close initially to the symmetry breaking point.
- Using perturbative analysis, we show in these solutions exist over the course of many periods of oscillation.
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Result Outline Cont.
Limitations of the Current Methods

▶ It should be noted our current theorem does not apply to perturbations of general periodic orbits described by the dynamical system.

▶ Though many of our techniques do generalize to this case, a better understanding of the period of oscillation from a dynamical systems point of view and the modulation techniques to control related growth in the Floquet map must be attained in order to analytically study such cases.

▶ It should be noted also that in the long term decay should occur to a soliton solution through coupling to the continuous spectrum as in Gang-Weinstein.
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One can slightly alter the ansatz above used to find solitons in order to allow the coefficients to depend on $t$, namely we take

$$u(x, t) = c_0(t)\psi_0 + c_1(t)\psi_1 + R(x, t).$$
Then, we have

\[
\begin{align*}
\dot{c}_0 - \Omega_0 c_0 + a_{0000} |c_0|^2 c_0 + a_{0011} (c_1^2 \bar{c}_0 + 2|c_1|^2 c_0) &= F_0(c_0, c_1, \bar{c}_0, \bar{c}_1; R, \bar{R}), \\
\dot{c}_1 - \Omega_1 c_1 + a_{1111} |c_1|^2 \rho_1 + a_{0011} (c_0^2 \bar{c}_1 + 2|c_0|^2 c_1) &= F_1(c_0, c_1, \bar{c}_0, \bar{c}_1; R, \bar{R}), \\
i \dot{R}_t - HR + F_\perp(c_0, c_1, \bar{c}_0, \bar{c}_1) &= G(c_0, c_1, \bar{c}_0, \bar{c}_1; R, \bar{R}).
\end{align*}
\]
Formulation as a Coupled Finite-Infinite Dimensional System Cont.

\[ a_{ijkl} = \langle \psi_i \psi_j \psi_k, \psi_l \rangle \]

\[ F_j = \langle 2|c_0|^2 \psi_0^2 + 2|c_1|^2 \psi_1^2 + 2(c_0 \bar{c}_1 + c_1 \bar{c}_0) \psi_0 \psi_1 \rangle R + \left[ c_0^2 \psi_0^2 + c_1^2 \psi_1^2 + 2c_0 c_1 \psi_0 \psi_1 \right] \bar{R} + [\bar{c}_1 \psi_1 + \bar{c}_0 \psi_0] R^2 + [2c_0 \psi_0 + 2c_1 \psi_1] |R|^2 + |R|^2 R, \psi_j \rangle \]

for \( j = 1, 2 \)
Formulation as a Coupled Finite-Infinite Dimensional System Cont.

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Formulation as a Coupled Finite-Infinite Dimensional System Cont.

\[ F_\perp = P_c \left[ |c_0|^2 c_0 \psi_0^3 + (c_0^2 \bar{c}_1 + 2|c_0|^2 \rho_1) \psi_0^2 \psi_1 
+ (c_1^2 \bar{c}_0 + 2c_0|c_1|^2) \psi_0 \psi_1^2 + |c_1|^2 c_1 \psi_1^3 \right], \]

\[ G = P_c \left( \left[ c_0^2 \psi_0^2 + c_1^2 \psi_1^2 + 2c_0 c_1 \psi_0 \psi_1 \right] \bar{R} 
+ \left[ 2|c_0|^2 \psi_0^2 + 2|c_1|^2 \psi_1^2 + 2(c_0 \bar{c}_1 + c_1 \bar{c}_0) \psi_0 \psi_1 \right] R 
+ [\bar{c}_1 \psi_1 + \bar{c}_0 \psi_0] R^2 + [2c_0 \psi_0 + 2c_1 \psi_1] |R|^2 
+ |R|^2 R \right). \]
Floquet Theory

- If there are periodic orbits associated to the coefficients in the leading order finite dimensional system, the operator associated with linearization about those orbits will be determined by a Floquet operator.

- There exists an example where the eigenvalues of a finite-dimensional, time-periodic matrix are all negative, suggesting asymptotic stability, where the solution blows up exponentially.

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- So, we must carefully understand that nature of the finite dimensionaly system before we proceed.
The ODE system

Let us look at

\[
\begin{align*}
    i \dot{\rho}_0 &= \Omega_0 \rho_0 - (\rho_0^2 \bar{\rho}_0 + 2 \rho_1 \bar{\rho}_1 \rho_0 + \rho_1^2 \bar{\rho}_0), \\
    i \dot{\bar{\rho}}_0 &= -\Omega_0 \bar{\rho}_0 + (\bar{\rho}_0^2 \rho_0 + 2 \rho_1 \bar{\rho}_1 \bar{\rho}_0 + \bar{\rho}_1^2 \rho_0), \\
    i \dot{\rho}_1 &= \Omega_1 \rho_1 - (\rho_1^2 \bar{\rho}_1 + 2 \rho_0 \bar{\rho}_0 \rho_1 + \rho_0^2 \bar{\rho}_1) \\
    i \dot{\bar{\rho}}_1 &= -\Omega_1 \bar{\rho}_1 + (\bar{\rho}_1^2 \rho_1 + 2 |\bar{\rho}_0|^2 \bar{\rho}_1 + \bar{\rho}_0^2 \rho_1).
\end{align*}
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For simplicity and without loss of generality we have taken \(a_{0000} = a_{0011} = a_{1111} = 1\).
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\end{align*}
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For simplicity and without loss of generality we have taken \( a_{0000} = a_{0011} = a_{1111} = 1 \).
The ODE system cont.

Inherent in this Hamiltonian system, we have the conserved quantities

\[
N = |\rho_0|^2 + |\rho_1|^2,
\]
\[
H = \Omega_0|\rho_0|^2 + \Omega_1|\rho_1|^2 - \frac{1}{2}|\rho_0|^4 - \frac{1}{2}|\rho_1|^4
\]
\[-2|\rho_1|^2|\rho_0|^2 - \frac{1}{2}(\rho_1^2\bar{\rho}_0^2 + \bar{\rho}_1^2\rho_0^2).
\]

A simple calculation shows
\[
 i\partial_t \tilde{\rho} = J \nabla_{\tilde{\rho}} H,
\]
where
\[
\tilde{\rho} = \begin{bmatrix}
\rho_0 \\
\bar{\rho}_0 \\
\rho_1 \\
\bar{\rho}_1
\end{bmatrix}, 
J = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
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\]
Hamiltonian Coordinates

Define

\[ \rho_0(t) = A(t)e^{i\theta(t)} \]

and

\[ \rho_1(t) = (\alpha(t) + i\beta(t))e^{i\theta(t)}. \]

Let

\[ u(x, t) = (A(t)e^{i\theta(t)}\psi_0 + (\alpha(t) + i\beta(t))e^{i\theta(t)}\psi_1) + R. \]
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Hamiltonian Coordinates Cont.

The system becomes

\[
\dot{A} = -2\alpha \beta A + \text{Error}_A(R, \bar{R}, \bar{\alpha}),
\]
\[
\dot{\alpha} = [\Omega_1 - \Omega_0 + 2\alpha^2] \beta + \text{Error}_\alpha(R, \bar{R}, \bar{\alpha}),
\]
\[
\dot{\beta} = -[\Omega_1 - \Omega_0 - 2A^2 + 2\alpha^2] \alpha + \text{Error}_\beta(R, \bar{R}, \bar{\alpha}),
\]
\[
\dot{\theta} = -\Omega_0 + A^2 + (3\alpha^2 + \beta^2) + \text{Error}_\theta(R, \bar{R}, \bar{\alpha})
\]
\[
iR_t - HR = F_\perp(A, \alpha, \beta, \theta) + G(A, \alpha, \beta, \theta; R, \bar{R}),
\]

The conservation laws for the finite dimensional part of the system become

\[
N = A^2 + \alpha^2 + \beta^2,
\]
\[
H = \Omega_0 A^2 + \Omega_1(\alpha^2 + \beta^2) - \frac{1}{2}(\alpha^2 + \beta^2)^2
\]
\[
- 2A^2(\alpha^2 + \beta^2) - A^2(\alpha^2 - \beta^2).
\]
Hamiltonian Coordinates Cont.

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\[ \dot{\theta} = -\Omega_0 + A^2 + (3\alpha^2 + \beta^2) + \text{Error}_\theta(R, \bar{R}, \bar{\alpha}) \]
\[ iR_t - HR = F_\perp(A, \alpha, \beta, \theta) + G(A, \alpha, \beta, \theta; R, \bar{R}), \]

The conservation laws for the finite dimensional part of the system become

\[ N = A^2 + \alpha^2 + \beta^2, \]
\[ H = \Omega_0 A^2 + \Omega_1(\alpha^2 + \beta^2) - \frac{1}{2}(\alpha^2 + \beta^2)^2 \]
\[ - 2A^2(\alpha^2 + \beta^2) - A^2(\alpha^2 - \beta^2). \]
Plugging this ansatz into the finite dimensional system and using the conservation laws we have

\[
\begin{align*}
\dot{\alpha} &= [\Omega_{10} + 2\alpha^2] \beta \\
\dot{\beta} &= -[\Omega_{10} + 2\beta^2 + 4\alpha^2 - 2N]\alpha \\
\dot{A} &= -2\alpha\beta A \\
\dot{\theta} &= -\Omega_0 + N + 2\alpha^2
\end{align*}
\]

where \(\Omega_{10} = \Omega_1 - \Omega_0\) and once again we have set \(a_{ijkl} = 1\) for all \(i, j, k, l = 0, 1\).
An analysis of the phase plane in the $(\alpha, \beta)$ coordinates shows the bifurcation point immediately as

$$N_{cr}^{FD} = \frac{\Omega_{10}}{2}$$

since it is the behavior of the dynamical system shifts above depending upon whether the first order term in $\dot{\beta}$ given by

$$-(\Omega_{10} - 2N)$$

is positive or negative.

For $N > \frac{\Omega_{10}}{2}$, there exist an equilibrium points away from $(\alpha, \beta) = (0, 0)$ and for $N < \frac{\Omega_{10}}{2}$ we have on the trivial equilibrium point. Note that for $N > \frac{\Omega_{10}}{2}$, the point $(0, 0)$ is still an equilibrium and in fact represents the singular point along the separatrix in the corresponding dynamical system.
Hamiltonian Coordinates Cont.

- An analysis of the phase plane in the \((\alpha, \beta)\) coordinates shows the bifurcation point immediately as

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Figure: A numerical plot of a phase plane with $\alpha$ plotted versus $\beta$ for $n > 0$. Here, we have $N_{cr} = .1$, $n = .05$. 
Figure: A numerical plot of a phase plane with $\alpha$ plotted versus $\beta$ for $n < 0$. Here, we have $N_{cr} = .1$, $n = -.05$. 
If we linearize about a particular periodic orbit

$$\vec{\alpha}_* = (\alpha_*, \beta_*, A_*, \theta_*)$$

with period $T_*$, then we have the equation

$$\partial_t \begin{bmatrix} \delta\alpha \\ \delta\beta \\ \delta A \\ \delta\theta \end{bmatrix} = B(t) \begin{bmatrix} \delta\alpha \\ \delta\beta \\ \delta A \\ \delta\theta \end{bmatrix}.$$
Hamiltonian Coordinates Cont.

- $B(t)$ is given by:

$$
\begin{bmatrix}
4\alpha_*\beta_* & \Omega_{10} + 2\alpha_*^2 & 0 & 0 \\
-\Omega_{10} - 2\beta_*^2 - 12\alpha_*^2 + 2N & -4\alpha_*\beta_* & 0 & 0 \\
-2A_*\beta_* & -2\alpha_*A_* & -2\alpha_*\beta_* & 0 \\
4\alpha_* & 0 & 0 & 0 \\
\end{bmatrix}
$$
Radial Coordinates

set $\rho_0 = r_0 e^{i\theta_0}$ and $\rho_1 = r_1 e^{i\theta_1}$. This leads to the following system of ODE’s:

\[
\begin{align*}
\dot{r}_0 &= r_1^2 r_0 \sin(2\Delta \theta), \\
\dot{r}_1 &= -r_0^2 r_1 \sin(2\Delta \theta), \\
(\Delta \theta) &= \Omega_1 - \Omega_0 + (r_1^2 - r_0^2)(1 + \cos(2\Delta \theta)),
\end{align*}
\]

where $\Delta \theta = \theta_1 - \theta_0$. 
Radial Coordinates Cont.

Define

\[ r_0 = \sqrt{\frac{\Omega_1 - \Omega_0}{2}} + \epsilon_0, \]
\[ r_1 = \epsilon_1, \]
\[ N = N_{\text{crit}} + n = \frac{\Omega_1 - \Omega_0}{2} + n, \]

where

\[ n = \epsilon_0^2 + \epsilon_1^2 + 2\sqrt{\frac{\Omega_1 - \Omega_0}{2}} \epsilon_0. \]
Radial Coordinates Cont.

Then, we have

\[\dot{\varepsilon}_0 = \varepsilon_1^2 \left( \sqrt{\frac{\Omega_1 - \Omega_0}{2}} + \varepsilon_0 \right) \sin(2\Delta \theta),\]

\[\dot{\varepsilon}_1 = -\varepsilon_1 \left( \sqrt{\frac{\Omega_1 - \Omega_0}{2}} + \varepsilon_0 \right)^2 \sin(2\Delta \theta),\]

\[(\dot{\Delta \theta}) = \Omega_1 - \Omega_0 \]

\[+ \left( \varepsilon_1^2 - \left( \sqrt{\frac{\Omega_1 - \Omega_0}{2}} + \varepsilon_0 \right)^2 \right) (1 + \cos(2\Delta \theta)).\]
Equilibrium Points in the Phase Diagram Cont.

Figure: A numerical plot of a phase plane with $\Delta \theta$ plotted versus $\epsilon_1$ for $n > 0$. Here, we have $N_{cr} = .2$, $n = .05$. 

Equilibrium Points in the Phase Diagram Cont.

Figure: A numerical plot of a phase plane with $\Delta \theta$ plotted versus $\epsilon_1$ for $n < 0$. Here, we have $N_{cr} = .2$, $n = -.05$. 
Linearization about a Finite Dimensional System

- We further refine our ansatz such that

\[
\begin{align*}
c_0(t) &= \rho_0(t) + \eta_0(t), \\
c_1(t) &= \rho_1(t) + \eta_1(t), \\
R(x, t) &= \tilde{R}(x, t) + w(x, t).
\end{align*}
\]

- We wish to linearize about an infinite dimensional solution, namely \( \rho_0, \rho_1 \) and \( \tilde{R} \) where

\[
i\tilde{R}_t - H\tilde{R} + F_\perp(\rho_0, \rho_1, \bar{\rho}_0, \bar{\rho}_1) = 0,
\]

or

\[
\begin{align*}
\tilde{R} &= -i \int_0^t e^{iH(t-s)} P_c F_\perp(\rho_0, \rho_1, \bar{\rho}_0, \bar{\rho}_1) ds \\
&= -i \int_0^t e^{iH(t-s)} F_\perp(\rho_0, \rho_1, \bar{\rho}_0, \bar{\rho}_1) ds.
\end{align*}
\]
Linearization about a Finite Dimensional System

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\begin{align*}
c_0(t) &= \rho_0(t) + \eta_0(t), \\
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i\tilde{R}_t - H\tilde{R} + F_\perp(\rho_0, \rho_1, \bar{\rho}_0, \bar{\rho}_1) = 0,
\]

or

\[
\tilde{R} = -i \int_0^t e^{iH(t-s)} P_c F_\perp(\rho_0, \rho_1, \bar{\rho}_0, \bar{\rho}_1) ds
\]

\[
= -i \int_0^t e^{iH(t-s)} F_\perp(\rho_0, \rho_1, \bar{\rho}_0, \bar{\rho}_1) ds.
\]
Linearization about a Finite Dimensional System Cont.

\[ i\dot{\eta}_0 - \Omega_0 \eta_0 + 2|\rho_0|^2 \eta_0 + \rho_0^2 \bar{\eta}_0 + \rho_1^2 \bar{\eta}_0 + 2\rho_0 \rho_1 \eta_1 + 2|\rho_1|^2 \eta_0 + 2 \rho_0 \rho_1 \bar{\eta}_1 + \rho_0 \bar{\rho}_1 \eta_1 = F_0(\rho_0 + \eta_0, \rho_1 + \eta_1, \bar{\rho}_0 + \bar{\eta}_0, \bar{\rho}_1 + \bar{\eta}_1; \tilde{R} + w, \tilde{R} + \tilde{w}) + \text{h.o.t.} \]

\[ i\dot{\eta}_1 - \Omega_1 \eta_1 + 2|\rho_1|^2 \eta_1 + \rho_0^2 \bar{\eta}_1 + \rho_1^2 \bar{\eta}_1 + 2\rho_1 \rho_0 \eta_0 + 2|\rho_0|^2 \eta_1 + 2 \rho_1 \rho_0 \bar{\eta}_0 + \rho_1 \bar{\rho}_1 \eta_0 = F_1(\rho_0 + \eta_0, \rho_1 + \eta_1, \bar{\rho}_0 + \bar{\eta}_0, \bar{\rho}_1 + \bar{\eta}_1; \tilde{R} + w, \tilde{R} + \tilde{w}) + \text{h.o.t.} \]
Linearization about a Finite Dimensional System Cont.

\[ i\dot{\eta}_0 - \Omega_0 \eta_0 + 2|\rho_0|^2\eta_0 + \rho_0^2\bar{\eta}_0 + \rho_1^2\bar{\eta}_0 + 2\rho_0\rho_1\eta_1 + 2|\rho_1|^2\eta_0 + 2\rho_0\rho_1\bar{\eta}_1 + \rho_0\rho_1\bar{\eta}_1 = F_0(\rho_0 + \eta_0, \rho_1 + \eta_1, \bar{\rho}_0 + \bar{\eta}_0, \bar{\rho}_1 + \bar{\eta}_1; \tilde{R} + w, \tilde{R} + \tilde{w}) + \text{h.o.t.} \]

\[ i\dot{\eta}_1 - \Omega_1 \eta_1 + 2|\rho_1|^2\eta_1 + \rho_1^2\bar{\eta}_1 + \rho_0^2\bar{\eta}_1 + 2\rho_1\rho_0\eta_0 + 2|\rho_0|^2\eta_1 + 2\rho_1\rho_0\bar{\eta}_0 + \rho_1\rho_0\bar{\eta}_0 = F_1(\rho_0 + \eta_0, \rho_1 + \eta_1, \bar{\rho}_0 + \bar{\eta}_0, \bar{\rho}_1 + \bar{\eta}_1; \tilde{R} + w, \tilde{R} + \tilde{w}) + \text{h.o.t.} \]
Linearization about a Finite Dimensional System Cont.

\[ i\omega t - H\omega = \]

\[ F_{\perp}(\rho_0, \rho_1, \bar{\rho}_0, \bar{\rho}_1) - F_{\perp}(\rho_0 + \eta_0, \rho_1 + \eta_1, \bar{\rho}_0 + \bar{\eta}_0, \bar{\rho}_1 + \bar{\eta}_1) \]

\[ + G(\rho_0 + \eta_0, \rho_1 + \eta_1, \bar{\rho}_0 + \bar{\eta}_0, \bar{\rho}_1 + \bar{\eta}_1; \tilde{R} + w, \tilde{\tilde{R}} + \tilde{\omega}), \]

where \( F_0, F_1 \) and \( G \) have higher order dependence on \( \tilde{R} + w \).
From the Hamiltonian structure of the equations for $\tilde{\rho}$, we see

$$i \dot{\tilde{\eta}} = JD_\rho^2 H \tilde{\eta} = B_{\tilde{\rho}}(t) H \tilde{\eta}.$$ 

As a result, we need bounds on the Floquet map for the periodic operator matrix $B_{\tilde{\rho}}(t)$. 
Linearization about a Finite Dimensional System Cont.

- From the Hamiltonian structure of the equations for $\tilde{\rho}$, we see

$$i\dot{\tilde{\eta}} = JD^2_{\rho} H \tilde{\eta}$$

$$= B_{\tilde{\rho}}(t) H \tilde{\eta}.$$ 

- As result, we need bounds on the Floquet map for the periodic operator matrix $B_{\tilde{\rho}}(t)$. 

We have

\[
\begin{aligned}
  i \dot{\eta} &= B \tilde{\rho}(t) \eta + F(\eta; \tilde{R} + w, \tilde{R} + \tilde{w}), \\
  iw_t - Hw + F_{\perp}(\rho + \eta, \tilde{\rho} + \tilde{\eta}) &= F_{\perp}(\rho, \tilde{\rho}) \\
  &= G(\rho_0 + \eta_0, \rho_1 + \eta_1, \tilde{\rho}_0 + \tilde{\eta}_0, \tilde{\rho}_1 + \tilde{\eta}_1; \tilde{R} + w, \tilde{R} + \tilde{w}),
\end{aligned}
\]

where

\[
|F_{\perp}(\rho + \eta, \tilde{\rho} + \tilde{\eta}) - F_{\perp}(\rho, \tilde{\rho})| \approx O(\eta).
\]
Perturbations of Equilibrium Solutions

In order to deal with a well understood solution operator, we first look at the equilibrium orbits such that \( n > 0 \) and \( \alpha = \sqrt{\frac{n}{2}} \) and \( \beta = 0 \).

The finite dimensional solution is of the form

\[
\vec{\alpha}_* (t) = (\alpha_*, \beta_*, A_*, \theta_*) (t) = \\
\left( \sqrt{\frac{n}{2}}, 0, \sqrt{N_{cr}} + \epsilon_{eq}^0, (-\Omega_0 + N_{cr} + 2n)t \right).
\]

Define the matrix \( B \) as

\[
\begin{bmatrix}
4\alpha_* \beta_* & (\Omega_{10} + 2\alpha_*^2) & 0 & 0 \\
-(\Omega_{10} + 6\alpha_*^2 - 2A_*^2) & 0 & -4\alpha_* A_* & 0 \\
-2A_* \beta_* & -2\alpha_* A_* & -2\alpha_* \beta_* & 0 \\
6\alpha_* & 2\beta_* & 2A_* & 0
\end{bmatrix},
\]

which is the matrix resulting from linearization about \( \vec{\alpha}_* \).
Perturbations of Equilibrium Solutions

- In order to deal with a well understood solution operator, we first look at the equilibrium orbits such that $n > 0$ and $\alpha = \sqrt{\frac{n}{2}}$ and $\beta = 0$.
- The finite dimensional solution is of the form

$$\tilde{\alpha}_*(t) = (\alpha_*, \beta_*, A_*, \theta_*)(t) = \left(\sqrt{\frac{n}{2}}, 0, \sqrt{N_{cr}} + \epsilon_{eq}^0, (-\Omega_0 + N_{cr} + 2n)t\right).$$

- Define the matrix $B$ as

$$B = \begin{bmatrix}
4\alpha_*\beta_* & (\Omega_{10} + 2\alpha_*^2) & 0 & 0 \\
-(\Omega_{10} + 6\alpha_*^2 - 2A_*^2) & 0 & -4\alpha_*A_* & 0 \\
-2A_*\beta_* & -2\alpha_*A_* & -2\alpha_*\beta_* & 0 \\
6\alpha_* & 2\beta_* & 2A_* & 0
\end{bmatrix},$$

which is the matrix resulting from linearization about $\tilde{\alpha}_*$.
In order to deal with a well understood solution operator, we first look at the equilibrium orbits such that \( n > 0 \) and \( \alpha = \sqrt{\frac{n}{2}} \) and \( \beta = 0 \).

The finite dimensional solution is of the form
\[
\tilde{\alpha}_*(t) = (\alpha_*, \beta_*, A_*, \theta_*)(t) = \\
(\sqrt{\frac{n}{2}}, 0, \sqrt{N_{cr}} + \epsilon_0^{eq}, (-\Omega_0 + N_{cr} + 2n)t).
\]

Define the matrix \( B \) as
\[
\begin{bmatrix}
4\alpha_*\beta_* & (\Omega_{10} + 2\alpha_*^2) & 0 & 0 \\
-\left(\Omega_{10} + 6\alpha_*^2 - 2A_*^2\right) & 0 & -4\alpha_*A_* & 0 \\
-2A_*\beta_* & -2\alpha_*A_* & -2\alpha_*\beta_* & 0 \\
6\alpha_* & 2\beta_* & 2A_* & 0 \\
\end{bmatrix},
\]
which is the matrix resulting from linearization about \( \tilde{\alpha}_* \).
Then, at equilibrium, we have

\[ B = \begin{bmatrix} * & * & * & 0 \\ * & \tilde{B}_{3 \times 3} & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 2N^{1/2} & 0 \end{bmatrix}. \]

showing that \( e^{Bt} \sim O(1 + N_{cr}^{1/2}t). \)
Then, at equilibrium, we have

\[
B = \begin{bmatrix}
* & * & * & 0 \\
* & \tilde{B}_{3 \times 3} & * & 0 \\
* & * & * & 0 \\
0 & 0 & 2N^{1/2} & 0
\end{bmatrix}.
\]

showing that \( e^{Bt} \sim O(1 + N_{cr}^{1/2}t) \).

\[
\tilde{B} = 2 \begin{bmatrix}
0 & N_{cr} & 0 \\
(N - N_{cr}) & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
Perturbations of Equilibrium Solutions Cont.

- $B$ has the same eigenvalues as $\tilde{B}$, with $\lambda = 0$ now a generalized eigenvalue of multiplicity two, implying linear growth of $\delta \theta(t)$. 
For the equilibrium orbits such that $n < 0$, we have $\alpha = 0$ and $\beta = 0$. The finite dimensional solution is of the form

$$\bar{\alpha}_*(t) = (\alpha_*, \beta_*, A_*, \theta_*)(t) = (0, 0, \sqrt{N_{cr} - \epsilon_0^{eq}}, (-\Omega_0 + N_{cr} - n)t)$$

and a similar decomposition to that above follows.
Hence we may state the following

**Proposition**

*Let $B$ be as above. Then, the equation*

$$\dot{\vec{y}} = B\vec{y}$$

*is given by*

$$e^{tB}\vec{y}_0,$$

*where*

$$|e^{tB}\vec{z}| \lesssim (1 + N_{cr}^{\frac{1}{2}}t)|\vec{z}|.$$
Let $\tilde{B}_\rho$ be the Floquet map associated to the system
\[ \dot{\eta} = B_\rho(t)\eta, \]
\[ \eta(0) = \eta_0. \]

Namely, the solution is of the form
\[ \eta = \tilde{B}_\rho(t)\eta_0. \]
Relation Between Linearization in $\tilde{\rho}$ and $\tilde{\alpha}$

Let $\tilde{B}_{\tilde{\rho}}$ be the Floquet map associated to the system

$$\dot{\eta} = B_{\tilde{\rho}}(t)\eta,$$

$$\eta(0) = \eta_0.$$

Namely, the solution is of the form

$$\tilde{\eta} = \tilde{B}_{\tilde{\rho}}(t)\eta_0.$$
Relation Between Linearization in \( \vec{\rho} \) and \( \vec{\alpha} \)

Cont.

- We wish to show for \( \vec{\rho} = \vec{\rho}^{eq} \) for some \( n \) as defined above implies that

\[
\| \tilde{B}_{\rho^{eq}}(t) \|_{L^\infty \to L^\infty} \lesssim (1 + N_{cr}^{\frac{1}{2}} t).
\]

- We have

\[
\begin{bmatrix}
\delta \rho_0 \\
\delta \rho_1 \\
\delta \bar{\rho}_0 \\
\delta \bar{\rho}_1
\end{bmatrix} = M(t) \begin{bmatrix}
\delta A \\
\delta \alpha \\
\delta \beta \\
\delta \theta
\end{bmatrix}.
\]
Relation Between Linearization in $\tilde{\rho}$ and $\tilde{\alpha}$

Cont.

- We wish to show for $\tilde{\rho} = \tilde{\rho}^{eq}$ for some $n$ as defined above implies that

$$\| \tilde{B}_{\tilde{\rho}^{eq}}(t) \|_{L^\infty \rightarrow L^\infty} \lesssim (1 + N_{Cr}^2 t).$$

- We have

$$\begin{bmatrix} \delta \rho_0 \\ \delta \rho_1 \\ \delta \bar{\rho}_0 \\ \delta \bar{\rho}_1 \end{bmatrix} = M(t) \begin{bmatrix} \delta A \\ \delta \alpha \\ \delta \beta \\ \delta \theta \end{bmatrix}.$$
Relation Between Linearization in $\vec{p}$ and $\vec{\alpha}$

Cont.

$$M(t) = \begin{bmatrix}
  e^{i\theta} & 0 & 0 & iAe^{i\theta} \\
  0 & e^{i\theta} & ie^{i\theta} & i(\alpha + i\beta)e^{i\theta} \\
  e^{-i\theta} & 0 & 0 & -iAe^{-i\theta} \\
  0 & e^{-i\theta} & -ie^{-i\theta} & -i(\alpha - i\beta)e^{-i\theta}
\end{bmatrix}$$
Relation Between Linearization in $\vec{\rho}$ and $\vec{\alpha}$

Cont.

» Since $M$ is invertible orbits near the equilibrium (at equilibrium $\det(M) = 4A^{eq} \neq 0$), we have

$$
\begin{bmatrix}
\delta A \\
\delta \alpha \\
\delta \beta \\
\delta \theta \\
\end{bmatrix} = M^{-1}(t) \begin{bmatrix}
\delta \rho_0 \\
\delta \rho_1 \\
\delta \bar{\rho}_0 \\
\delta \bar{\rho}_1 \\
\end{bmatrix}.
$$

» Then,

$$
\partial_t \begin{bmatrix}
\delta \rho_0 \\
\delta \rho_1 \\
\delta \bar{\rho}_0 \\
\delta \bar{\rho}_1 \\
\end{bmatrix} = (M_tM^{-1} + MBM^{-1}) \begin{bmatrix}
\delta \rho_0 \\
\delta \rho_1 \\
\delta \bar{\rho}_0 \\
\delta \bar{\rho}_1 \\
\end{bmatrix}.
$$
Relation Between Linearization in $\vec{\rho}$ and $\vec{\alpha}$

Cont.

- Since $M$ is invertible orbits near the equilibrium (at equilibrium $\det(M) = 4A^{eq} \neq 0$), we have

\[
\begin{bmatrix}
\delta A \\
\delta \alpha \\
\delta \beta \\
\delta \theta
\end{bmatrix}
= M^{-1}(t)
\begin{bmatrix}
\delta \rho_0 \\
\delta \rho_1 \\
\delta \bar{\rho}_0 \\
\delta \bar{\rho}_1
\end{bmatrix}.
\]

- Then,

\[
\partial_t \begin{bmatrix}
\delta \rho_0 \\
\delta \rho_1 \\
\delta \bar{\rho}_0 \\
\delta \bar{\rho}_1
\end{bmatrix}
= (M_t M^{-1} + MBM^{-1})
\begin{bmatrix}
\delta \rho_0 \\
\delta \rho_1 \\
\delta \bar{\rho}_0 \\
\delta \bar{\rho}_1
\end{bmatrix}.
\]
Relation Between Linearization in $\vec{\rho}$ and $\vec{\alpha}$ Cont.

- At equilibrium for $n > 0$, we have:

$$M(t) =$$

$$\begin{bmatrix}
  e^{i\kappa t} & 0 & 0 & iA^{eq} e^{i\kappa t} \\
  0 & e^{i\kappa t} & ie^{i\kappa t} & i\alpha^{eq} e^{i\kappa t} \\
  e^{-i\beta t} & 0 & 0 & -iA^{eq} e^{-i\beta t} \\
  0 & e^{-i\kappa t} & -ie^{-i\kappa t} & -i\alpha^{eq} e^{-i\kappa t}
\end{bmatrix}$$

for

$$\kappa = (-\Omega_0 + N_{cr} + 2n).$$
Relation Between Linearization in $\vec{\rho}$ and $\vec{\alpha}$

Cont.

- At equilibrium for $n > 0$, we have:

\[
\mathcal{M}(t) =
\begin{bmatrix}
  e^{i\kappa t} & 0 & 0 & iA^{eq} e^{i\kappa t} \\
  0 & e^{i\kappa t} & ie^{i\kappa t} & i\alpha^{eq} e^{i\kappa t} \\
  e^{-i\beta t} & 0 & 0 & -iA^{eq} e^{-i\beta t} \\
  0 & e^{-i\kappa t} & -ie^{-i\kappa t} & -i\alpha^{eq} e^{-i\kappa t}
\end{bmatrix}
\]

for $\kappa = (-\Omega_0 + N_{cr} + 2n)$. 

Relation Between Linearization in $\vec{\rho}$ and $\vec{\alpha}$ Cont.

\[ M_t(t) = \kappa \times \]

\[
\begin{bmatrix}
  e^{i\kappa t} & 0 & 0 & iA^{eq} e^{i\kappa t} \\
  0 & e^{i\kappa t} & ie^{i\kappa t} & i\alpha^{eq} e^{i\kappa t} \\
  -e^{-i\kappa t} & 0 & 0 & iA^{eq} e^{-i\kappa t} \\
  0 & -e^{-i\kappa t} & ie^{-i\kappa t} & i\alpha^{eq} e^{-i\kappa t}
\end{bmatrix}.
\]
Relation Between Linearization in $\vec{\rho}$ and $\vec{\alpha}$

Cont.

- So,

$$M_t M^{-1} = i\kappa \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

- Hence, we have

$$B_{\vec{\rho}}(t) = i\kappa \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + M(t)B M^{-1}(t)$$

$$= iQ(n) + M(t)B M^{-1}(t).$$
Relation Between Linearization in $\vec{ρ}$ and $\vec{α}$

Cont.

- So,

$$M_tM^{-1} = i\kappa \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

- Hence, we have

$$B_{\vec{ρ}}(t) = i\kappa \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + M(t)BM^{-1}(t)$$

$$= iQ(n) + M(t)BM^{-1}(t).$$
Relation Between Linearization in $\vec{\rho}$ and $\vec{\alpha}$

Cont.

- By computation, we know that
  \[
  \dot{\vec{y}} = B\vec{y},
  \]
  \[
  \vec{y}(0) = \vec{y}_0
  \]
  gives a bounded solution of the form
  \[
  \vec{y} = e^{Bt}\vec{y}_0.
  \]

- Here, we have simply made a Floquet map change of variables and see that the above equation results from setting
  \[
  \vec{\eta} = M(t)\vec{y}.
  \]

- Hence, using the boundedness of the matrix operator $M$, we obtain the desired bound
  \[
  \|\tilde{B}_{\vec{\rho}}(t)\|_{L^\infty \rightarrow L^\infty} \lesssim (1 + N_{cr}t).
  \]
Relation Between Linearization in $\vec{\rho}$ and $\vec{\alpha}$

Cont.

By computation, we know that

$$\dot{\vec{y}} = B\vec{y}$$
$$\vec{y}(0) = \vec{y}_0$$

gives a bounded solution of the form

$$\vec{y} = e^{Bt}\vec{y}_0.$$  

Here, we have simply made a Floquet map change of variables and see that the above equation results from setting

$$\vec{\eta} = M(t)\dot{\vec{y}}.$$  

Hence, using the boundedness of the matrix operator $M$, we obtain the desired bound

$$\|\tilde{B}_\rho(t)\|_{L^\infty \to L^\infty} \lesssim (1 + N_{cr}t).$$
Relation Between Linearization in $\vec{\rho}$ and $\vec{\alpha}$

Cont.

- By computation, we know that

$$\dot{\vec{y}} = B\vec{y}$$

$$\vec{y}(0) = \vec{y}_0$$

gives a bounded solution of the form

$$\vec{y} = e^{Bt}\vec{y}_0.$$  

- Here, we have simply made a Floquet map change of variables and see that the above equation results from setting

$$\vec{\eta} = M(t)\dot{\vec{y}}.$$

- Hence, using the boundedness of the matrix operator $M$, we obtain the desired bound

$$\|\tilde{B}_\rho(t)\|_{L^\infty \to L^\infty} \lesssim (1 + N_{cr}t).$$
Relation Between Linearization in $\vec{\rho}$ and $\vec{\alpha}$

Cont.

- The case for $n < 0$ is treated similarly. Most importantly, in order to understand Floquet multipliers, one looks for a periodic, well-defined change of variables which brings the time dependent system to a constant coefficient system. We are given that for free with $M$. 
Periodic Orbits

- Linearizing about the equilibrium solutions for \( n > 0 \), one may define

\[
\alpha = \frac{\sqrt{n}}{2} + h_1, \\
\beta = h_2,
\]

where \( h_1, h_2 \) small perturbations.

- Plugging in this ansatz gives

\[
\ddot{h}_1 = -4n \left( N_{cr} + \frac{n}{2} \right) h_1 + O(|\vec{h}|^2).
\]

- Hence, the period of oscillation near the equilibrium point is of the form

\[
T^*(n) = \frac{\pi}{n^{\frac{1}{2}} (N_{cr} + n)^{\frac{1}{2}}}.
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$$T^*(n) = \frac{\pi}{n^{1/2}(N_{cr} + n)^{1/2}}.$$
Similarly, for \( n < 0 \) we have

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\alpha = h_1, \\
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Hence, the period of oscillation near the equilibrium point is of the form

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Proposition

Fix $n > 0$ or $n < 0$ such that $|n| \ll 1$. Let us assume $N_{cr} \gg |n|$. Take $(\rho_0^{eq}, \rho_1^{eq})$ be the corresponding equilibrium solution. For any periodic solution $\rho_0(t), \rho_1(t)$ such that

$$|(\rho_0(t), \rho_1(t)) - (\rho_0^{eq}, \rho_1^{eq})| \ll n,$$

we have

$$(\rho_0(t + T), \rho_1(t + T)) = (\rho_0(t), \rho_1(t))$$

where

$$|T - \frac{\pi}{(|n|N_{cr})^{\frac{1}{2}}}| \ll n^2.$$
Remark

For small perturbations of the equilibrium point (for either \( n < 0 \) or \( n > 0 \)), it is for precisely the period \( T^*(n) \) on which we must control the coupling to the continuous spectrum for the full solution in order to prove these finite dimensional structures are observable over many oscillations. In order to generalize our result to any periodic solution predicted by the finite dimensional dynamics, we must understand fully the period of each full oscillation.
Numerical Experiments

**Figure:** A numerical plot of the symmetry breaking in the soliton curve. Specifically, we plot here $N$ versus $\Omega$ for a double Gaussian potential with $L = 3$ and $\sigma = 1$. 
Numerical Experiments

Figure: A numerical plot of the phase plane diagram for $n > 0$ with specific points chosen along a closed orbit which shows full oscillation of mass from one well to another.
Figure: A numerical plot of the absolute value of the solution to Equation (1.1) at various times with initial data such that $n > 0$ and $\Delta \theta(0) = 1$. The plots correspond to points a, b, c, d, e, and f respectively from the specified orbit in 8.
Figure: A numerical plot of the phase plane diagram for $n > 0$ with specific points chosen along a closed orbit which shows localization of the mass on one side of the well.
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Numerical Experiments

Figure: A numerical plot of the phase plane diagram for $n < 0$ with specific points chosen along a closed orbit.
Numerical Experiments

Figure: A numerical plot of the absolute value of the solution to Equation (1.1) at various times with initial data such that $n < 0$ and $\Delta \theta(0) = 0$. The plots correspond to points a, b, c, d, e, and f respectively from the specified orbit in 12.
Numerical Experiments

Figure: A numerical plot of the maximum amplitude of an oscillatory solution decaying to a ground state.
Figure: A numerical plot of the center of mass of an oscillatory solution decaying to a ground state.
Singular Potentials

The operator

\[ H_{q,L} = -\Delta - q(\delta(x + L) + \delta(x - L)) \]

has for \( L \) large enough unique symmetric and asymmetric bound states depending upon \( q \) and \( L \) one can explicitly construct.

It has a well-defined scattering matrix and hence the spectrum can be well-understood.

The results from Weder depend only on \( L^1 \) bounds on the potential to get essentially resolvent estimates. When done carefully, this can be done in the case above.
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Synopsis and Suggestions for Future Work

- Describing stability about generic oscillations and collapse to a ground state, generalizing this result to motion amongst many wells and eventually finding the Peireles-Nabbaro barrier for motion between wells or nodes in the discrete limit are all future applications of this work.

- The fact that we may observe oscillations between potential wells in systems such is not a new result, however we have been able to give a representation of the phenomenon in terms of a classical oscillatory system.
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- In future work on double well potentials, the authors hope to prove long time stability for oscillations far from the equilibrium point and optimize the time of existence proof by having better control of the damping caused by coupling to the continuous spectrum.

- In addition, these pseudo-bound states represent possible solutions resulting from the problem of scattering of solitons from $\infty$ across double well potential wells.
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Finally, the authors would like to point out this question of oscillation and the resulting dynamical systems becomes more challenging and interesting as the number of wells is increased.

In particular, as one increases the number of wells to ∞, the phase shift required to see oscillation from one well to the next might give insight into the celebrated Peireles-Nabbaro barrier for discrete nonlinear Schrödinger systems.
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