Mathematics of the quantum many–body problem and condensed matter physics

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Outline

1. Quantum Many-body problem and variational problems
2. Quantum Many-body problem and stochastic process
3. Quantum Many-body problem and analysis
4. Quantum Many-body problem and $C^*$-algebras
5. Quantum Many-body problem and differential equations
6. Concluding remarks
Bose-Einstein condensation (BEC)

- **BEC** is formally a new state of matter, where QM wave functions of atoms behave as **coherent matter waves** in the same way as coherent light waves in the case of a laser.

- A fraction of BEC in superfluid $^4$He was found in the sixties via deep inelastic neutron scattering.

- The theory of $^4$He is based on the **Bogoliubov-Landau theory**.

- **Pure** BEC was first undertaken in ’95 for dilute ultra-cold gases (Nobel Prize in ’01).

- BEC in dilute ultra–cold gases are described via the one-body **Gross-Pitaevskii equation**.
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Quantum Many-body problem and variational problems

Bose gases in canonical ensemble

**QM Hamiltonian**

QM Hamiltonian in the rotating frame for $N$ interacting bosons:

$$H_N = \sum_{j=1}^{N} \left\{ \left[ -i \nabla_j + \vec{A}_\Omega(\vec{x}_j) \right]^2 + V(\vec{x}_j) - \frac{\Omega^2 r_j^2}{4} \right\} + \sum_{1 \leq i < j \leq N} v(|\vec{x}_i - \vec{x}_j|)$$

- $\vec{x}_j \in \mathbb{R}^3$, $j = 1, \ldots, N$ and $r_j = |\hat{e}_z \wedge \vec{x}_j|$.
- $H_N$ acts on the Hilbert space $\mathcal{H}_+^{(N)}$ of **symmetric** wave functions in $L^2(\mathbb{R}^{3N}, d\vec{x}_1 \cdots d\vec{x}_N)$.
- $V(\vec{x})$ external trapping potential.
- $v(|\vec{x} - \vec{y}|) \geq 0$ pair interaction.
- $\Omega$ angular velocity (along the $z-$axis).
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At temperature $\beta^{-1} > 0$, its thermodynamics is given via:

**QM free-energy density and Gibbs state (non-zero temperature)**

$$f_N := -\frac{1}{\beta N} \ln \text{Trace}_{\mathcal{H}_+^{(N)}} \left( e^{-\beta H_N} \right), \quad \omega_N (\cdot) := \frac{\text{Trace}_{\mathcal{H}_+^{(N)}} (\cdot e^{-\beta H_N})}{\text{Trace}_{\mathcal{H}_+^{(N)}} (e^{-\beta H_N})}$$
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\]

**QM Energy and ground state (zero-temperature)**

\[
 E_N := \inf_{\Phi \in \mathcal{H}_+^{(N)}, ||\Phi||_2=1} \langle \Phi, H_N \Phi \rangle = \lim_{n \to \infty} \langle \Phi_n^{(N)}, H_N \Phi_n^{(N)} \rangle
\]

is the ground state energy of $H_N$ and $\{\Phi_n^{(N)}\}_n$ are approximate ground states. A ground state is a solution of this variational problem.

For macroscopic systems, one performs the limit where $N \to \infty$. 
Example (Thomas-Fermi (TF) Limit for rotating dilute gases)

1. The external trapping potential \( V \) satisfies: \( V(\lambda \vec{x}) = \lambda^s V(\vec{x}) \).
2. \( \nu(|\vec{x}|) = a^{-2} \nu_1(a^{-1}|\vec{x}|) \) with scattering length \( a \) depending on \( N \).
3. The angular velocity (along the \( z \)-axis) \( \Omega \) is also depending on \( N \).

Then, the TF limit means that the GP parameter \( g := 4\pi Na \to \infty \) as \( N \to \infty \), but the gas is still dilute (mean interparticle distance \( \gg \) range of interactions characterized by \( a \)).

Recall:

\[
H_N = \sum_{j=1}^{N} \left\{ \left[ -i\nabla_j + \vec{A}_\Omega(\vec{x}_j) \right]^2 + V(\vec{x}_j) - \frac{\Omega^2 r_j^2}{4} \right\} + \sum_{1 \leq i < j \leq N} \nu(|\vec{x}_i - \vec{x}_j|)
\]

with QM ground state energy (zero-temperature)

\[
E_N := \inf_{\Phi \in \mathcal{H}_+^{(N)}, \|\Phi\|_2 = 1} \langle \Phi, H_N \Phi \rangle
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Example (Thomas-Fermi (TF) Limit for rotating dilute gases)

1. The external trapping potential $V$ satisfies: $V(\lambda \vec{x}) = \lambda^s V(\vec{x})$.

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Define:

- $\Psi_N \in L^2(\mathbb{R}^{3N})$ approximate ground state, i.e., $\lim_{N \to \infty} \frac{\langle \Psi_N, H_N \Psi_N \rangle}{E_N} = 1$

- $\rho_N (\vec{x}) = \int_{\mathbb{R}^{3(N-1)}} d\vec{x}_2 \ldots d\vec{x}_N |\Psi_N (\vec{x}, \vec{x}_2, \ldots, \vec{x}_N)|^2$

**Main Goal in the TF limit $N, g, \Omega \to \infty$:**

$$\lim_{N \to \infty} \frac{E_N}{N} = ? \quad \rho_N \to ?$$
Theorem (QM Asymptotics, TF limit, [BCPY '08])

Let $V$ be a homogenous potential of order $s > 2$, define $g = 4\pi a N$ and
$\omega = g^{-\frac{s-2}{2(s+3)}} \Omega$, and assume that $N^{-2} g^3 \| \rho_{g,\Omega}^{\text{TF}} \|_\infty \to 0$ as $N \to \infty$.

1. **[Slow/Rapid Rotation]** If $g \to \infty$ and $\omega \geq 0$ is fixed as $N \to \infty$, then

$$\lim_{N \to \infty} \left\{ g^{-\frac{s}{s+3}} \frac{E_N}{N} \right\} = E_{1,\omega}^{\text{TF}}, \quad g^{\frac{3}{s+3}} \rho_N \left( g^{\frac{1}{s+3}} \vec{x} \right) \to \rho_{1,\omega}^{\text{TF}}(\vec{x})$$

and in weak $L^1$-sense.

2. **[Ultrarapid Rotation]** If $\Omega \to \infty$ and $\omega \to \infty$ as $N \to \infty$, then

$$\lim_{N \to \infty} \left\{ \Omega^{-\frac{2s}{s-2}} \frac{E_N}{N} \right\} = E_{0,1}^{\text{TF}}, \quad \Omega^{6/(s-2)} \rho_N \left( \Omega^{2/(s-2)} \vec{x} \right)$$

converges to a measure supported on the set $\mathcal{M}$ of minima of the function $V(\vec{x}) - \frac{1}{4} r^2$.

Quantum Many-body problem and variational problems

**TF Energy energy and ground state density (one-body problem)**

\[ E_{g,\omega}^{\text{TF}} := \inf_{\|\rho\| = 1} \left\{ \int_{\mathbb{R}^3} d\vec{x} (V\rho - \frac{\omega^2 r^2 \rho}{4} + g\rho^2) \right\} \]

and the unique solution of this variational problem is the function \( \rho_{g,\omega}^{\text{TF}}(\vec{x}) \).

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**Figure:** Different stages of giant vortex formation process.

- (a) Starting point: BEC after evaporative spin-up.
- (b)-(h) Laser shone onto BEC for (b) 14 s, (c) 15 s, (d) 20 s, (e) 22 s, (f) 23 s, (g) 40 s, (h) 70 s. Pictures are taken after 5.7-fold expansion of the BEC. Laser power is 8 fW.
- (i) Zoomed-in core region of (f).
Bose gas at non-zero temperature

One needs to compute $f_N := -\frac{1}{\beta N} \ln \text{Trace}_{\mathcal{H}_+^{(N)}} (e^{-\beta H_N})$ as $N \to \infty$. 
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Let \( \Omega = 0 \). Via the Feynman-Kac formula,

\[
\text{Trace}_{\mathcal{H}_+^{(N)}} \left( e^{-\beta H_N} \right) = \frac{1}{N!} \sum_{\sigma \in S_N} \int_{\mathbb{R}^{3N}} d\vec{x}_1 \cdots d\vec{x}_N \int_{C_N} \left( \bigotimes_{i=1}^N \nu_\beta^{\vec{x}_i, \vec{x}_{\sigma(i)}} \right) (db)
\]

\[
\times \exp \left\{ -\sum_{i=1}^N \int_0^\beta V(b_s^{(i)}) ds - \sum_{1 \leq i < j \leq N} \int_0^\beta \nu(||b_s^{(i)} - b_s^{(j)}||) ds \right\}
\]

- \( B \) is a Brownian motions having generator \( \Delta \). We write \( P_{\vec{x}} \) for the probability measure under which \( B \) starts from \( \vec{x} \in \mathbb{R}^3 \).
- \( b = (b_s^{(1)}, \ldots, b_s^{(N)}) \) are \( N \) Brownian bridges with measure on the time interval \([0, \beta] \) with initial site \( \vec{x} \in \mathbb{R}^3 \) and terminal site \( \vec{y} \in \mathbb{R}^3 \) is defined as

\[
\nu_\beta^{\vec{x}, \vec{y}}(A) = \frac{P_{\vec{x}} (B \in A, B_\beta \in d\vec{y})}{d\vec{y}}, \quad A \subset C \left( [0, \beta], \mathbb{R}^3 \right)
\]
A study of $f_N$ as $N \to \infty$ requires techniques coming from combinatorics, the theories of stochastic processes and large deviations.

Recent results in this direction (only for $\nu = 0$) is given by


These works are a continuation of the following papers:

   
   Large deviations as $\beta \to \infty$ for trapped interacting Brownian particles and paths. In particular, we invent a path interaction model related to the so-called Hartree approximation in physics.

   
   Large deviation as $N \to \infty$ in the Gross-Pitaevskii (GP) limit ($g := 4\pi Na$ fixed, cf. dilute gases) of path-repellent Brownian motions in a trap at $\beta^{-1} > 0$. We obtain a $\beta$-dependent of the GP variational problem.
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Bose gases in grand-canonical ensemble

\( \Omega = V = 0 \) and bosons are enclosed in a cubic box \( \Lambda \subset \mathbb{R}^3 \) of length \( L \).

**QM Hamiltonian by assuming periodic boundary conditions**

\[
H_\Lambda := \sum_{k \in \Lambda^*} k^2 a_k^* a_k - \mu N_\Lambda + \frac{1}{2L^3} \sum_{k_1, k_2, q \in \Lambda^*} \hat{\nu}(q) a_{k_1+q}^* a_{k_2-q} a_{k_1} a_{k_2}
\]

- \( H_\Lambda \) acts on the boson Fock space \( \mathcal{F}_{\Lambda,+} := \bigoplus_{N=0}^{\infty} \mathcal{H}^{(N)}_+; H_\Lambda|_{\mathcal{H}^{(N)}_+} = H_N \).
- \( a_k, a_k^* \) are the annihilation/creation operators of a boson in the state
  \( \psi_k(x) := L^{-3/2} e^{i k x}, \ k \in \Lambda^* := \left( \frac{2\pi}{L} \mathbb{Z} \right)^3, \ x \in \Lambda. \)
- The chemical potential \( \mu \) is the lagrange parameter associated with the particle number operator \( N_\Lambda := \sum_{k \in \Lambda^*} a_k^* a_k. \)
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\]

At temperature \( \beta^{-1} > 0 \), its thermodynamic is given via:

**QM pressure and Gibbs states (non-zero temperature)**

\[
p_\Lambda := \frac{1}{\beta L^3} \ln \text{Trace}_{\mathcal{F}_{\Lambda,+}} \left( e^{-\beta H_\Lambda} \right), \quad \omega_\Lambda (\cdot) := \frac{\text{Trace}_{\mathcal{F}_{\Lambda,+}} \left( \cdot e^{-\beta H_\Lambda} \right)}{\text{Trace}_{\mathcal{F}_{\Lambda,+}} (e^{-\beta H_\Lambda})}
\]
Thermodynamic properties of the many-body problem are completely unsolved except for trivial choices of the interaction $v$.

To explain quantum phases, one uses simplified models.

Example (The celebrated Bogoliubov superfluidity theory of helium)

This theory (’47) is based on a truncation of $H_\Lambda$ by assuming BEC.

1. We mathematically solved this model in the limit $L \to \infty$. Bogoliubov assumptions are mathematically inexact.

2. We demonstrate the existence of a new kind of BEC. All together, it corresponds to 7 papers with V. Zagrebnov and B. Nachtergaele (’98 -’02) and a Physics Reports of 143 pages (’01).

3. With S. Adams, we proposed a new superfluidity theory for non-dilute gases, mathematically solved as $L \to \infty$ (5 papers: Annales Henri Poincaré…, ’04 -’08), which implies new quantum phenomena.
Superconductivity

I. Historical overview:

- 1911: Discovery of mercury superconductivity \( T_c \leq 4.2 \) K.
- 1957: BCS theory for conventional type I superconductors \( \max\{T_c\} \leq 39 \) K.
- 1986: High–\( T_c \) superconductors in a ceramics material \( \max\{T_c\} \leq 200 \) K.

II. Superconducting materials: usual metals, magnetic materials, heavy–fermions systems, organic compounds, ceramics.

III. Main physical properties:

- Zero–resistivity below \( T_c \).
- Meißner–Ochsenfeld effect (except magnetic superconductors).
Electron gases on a lattice

- Electrons are enclosed in $\Lambda := \{\mathbb{Z} \cap [-L, L]\}^{d \geq 1}$ with volume $|\Lambda|$.
- The electron Fock space $\mathcal{F}_{\Lambda,+}$ is the infinite direct sum of antisymmetric $L_2$-functions on $\Lambda$.
- The operator $a_{x,s}^*$ resp. $a_{x,s}$ creates resp. annihilates an electron with spin $s \in \{\uparrow, \downarrow\}$ at $x \in \mathbb{Z}^d$. They satisfy the CAR:
  \[ a_{x,s}^* a_{y,s'}^* + a_{y,s'}^* a_{x,s}^* = 0 \]
  \[ a_{x,s} a_{y,s'} + a_{y,s'} a_{x,s} = 0 \]
  \[ a_{x,s} a_{y,s'}^* + a_{y,s'}^* a_{x,s} = \delta_{x,y} \delta_{s,s'} \]

  They also generate a $C^*$-algebra $\mathcal{U}$.

The remarkable degree of universality of phase transitions allows us to focus on simple models via thermodynamic analyses ($L \to \infty$).
Example (The strong coupling BCS-Hubbard model [BP’09])

\[ H_\Lambda := -\mu \sum_{x \in \Lambda} (n_{x,\uparrow} + n_{x,\downarrow}) - h \sum_{x \in \Lambda} (n_{x,\uparrow} - n_{x,\downarrow}) + 2\lambda \sum_{x \in \Lambda} n_{x,\uparrow} n_{x,\downarrow} - \frac{\gamma}{|\Lambda|} \sum_{x,y \in \Lambda} a_{x,\uparrow}^* a_{x,\downarrow}^* a_y, \downarrow a_y, \uparrow \]

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At non-zero temperature $\beta^{-1} > 0$, its thermodynamics is given via:

**Strong coupling BCS-Hubbard pressure and Gibbs states**

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p_{\Lambda} := \frac{1}{\beta |\Lambda|} \ln \text{Trace} \left( e^{-\beta H_{\Lambda}} \right), \quad \omega_{\Lambda}(\cdot) := \frac{\text{Trace} \left( \cdot e^{-\beta H_{\Lambda}} \right)}{\text{Trace} \left( e^{-\beta H_{\Lambda}} \right)}
\]

**Remark 1:**

1. The strong coupling BCS-Hubbard $H_{\Lambda}$ is a local element of the $C^*$-algebra $\mathcal{U}$ generated by $a_{x,s}^*, a_{x,s}$ for $s \in \{\uparrow, \downarrow\}$ and $x \in \mathbb{Z}^d$.

2. A state $\omega : \mathcal{U} \to \mathbb{C}$ is a positive and normalized linear functional on the $C^*$-algebra $\mathcal{U}$. An example is given by $\omega_{\Lambda}$ by periodically extending it.

3. The symmetry of $H_{\Lambda}$ (here permutation invariance inside $\Lambda$) implies the same symmetry on the local state $\omega_{\Lambda}$.

4. As $L \to \infty$, $\omega_{\Lambda}$ should converge to a permutation invariant state.
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**Remark 2:**

1. We define $E$ to be the set of all permutation invariant states, i.e.,

$$E := \left\{ \text{all states } \omega : \text{such that for any } \pi \in \Pi, \, \omega \circ \eta_{\pi} = \omega \right\}$$

Here the set $\Pi$ of bijective maps from $\mathbb{N}$ to $\mathbb{N}$, which leaves all but finite elements invariant, defines a group homomorphism $\eta : \Pi \to \text{Aut}(\mathcal{U})$ via $\kappa : \mathbb{N} \to \mathbb{Z}^d$: $\eta_{\pi} : a_{\kappa}(\ell),s \mapsto a_{\kappa(\pi(\ell))},s$ for $\pi \in \Pi, \, \ell \in \mathbb{N}, \, s \in \uparrow, \downarrow$.

2. $E$ is weak-$*$ compact, convex, and metrizable. By Choquet theorem, it has a unique decomposition in terms of extremal states.
Theorem (Pressure [BP’09])

For any \( \beta, \gamma > 0 \) and \( \mu, \lambda, h \in \mathbb{R} \),

\[
\lim_{L \to \infty} \{ p(L, \beta, \mu, \lambda, \gamma, h) \} = - \inf_{\omega \in E} \mathcal{F}(\omega) = \beta^{-1} \ln 2 + \mu + \sup_{r \geq 0} f(r)
\]

with \( \omega \mapsto \mathcal{F}(\omega) \) being affine and lower semicontinuous (weak-* topology) and \( f \in C[\mathbb{R}^+, \mathbb{R}] \) being an explicit function.

The set of equilibrium states is defined to be the non empty face \( \Omega_\beta \in E \) of minimizers of \( \mathcal{F}(\omega) \). (\( E \) is a weak-* compact and convex set).

Theorem (Gibbs state [BP’09])

Under assumptions of the previous theorem, the local Gibbs states \( \omega_L \) converges in the weak-* topology to an equilibrium states (which could be explicitly characterized).

Left figure: Illustration of the critical temperature $\theta_c(\lambda)$ for $\gamma = 2.6$, $h = 0$, $\mu = 1.25$, where superconductivity appears.
1. Left figure: Illustration of the critical temperature $\theta_c(\lambda)$ for $\gamma = 2.6$, $h = 0$, $\mu = 1.25$, where superconductivity appears.

2. Figure on the center. The red, green, and blue lines are respectively the Cooper pair condensate density $r_\beta$, the magnetization density $m_\beta$ and the electron density $d_\beta$ as functions of $\mu$ for $\beta = 30$, $\lambda = 0.575$, $\gamma = 2.6$, and $h = 0.1$. 
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3. Illustration of the critical temperature $\theta_c(\mu)$ for $\lambda = 0.575$, $\gamma = 2.6$, and $h = 0.1$. The black dashed line corresponds to half-filling.
Flow equations for operators

1. Let the unitary operator $U_{t,s}$ on a Hilbert space $\mathcal{H}$ be the solution of the non-autonomous evolution equation

$$\forall t \geq s \geq 0 : \quad \partial_t U_{t,s} = -i G_t U_{t,s}, \quad U_{s,s} := 1,$$

with self-adjoint (s.a.) generator $G_t$. Here $U_t := U_{t,0}$.
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2. Let $H_0 = H_0^*$ acting on $\mathcal{H}$. Then $H_t := U_t H_0 U_t^*$ satisfies

$$\forall t \geq 0 : \quad \partial_t H_t = i [H_t, G_t] := i(H_t G_t - G_t H_t), \quad H_{t=0} := H_0.$$
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3. Question: find $G_t$ depending on a fixed operator $A$ such that in the limit $t \to \infty$, $H_\infty = U_\infty H_0 U_\infty^*$ with

$$[A, H_\infty] := AH_\infty - H_\infty A = 0.$$
Assume that $H_0 = H_0^*$ and $A = A^*$ are two self-adjoint matrices.

Let $f(t) := \text{Trace}((H_t - A)^2) \geq 0$ and observe that

$$
\partial_t f(t) = \partial_t \left\{ \text{Trace} \left( H_t^2 - 2H_tA + A^2 \right) \right\} = \partial_t \left\{ \text{Trace} \left( -2H_tA \right) \right\} \\
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**Choice of the generator**

$$
G_t := i [A, H_t] := i (AH_t - H_tA)
$$

1. We then obtain $\partial_t f(t) = -2 \text{Trace} (G_t G_t^*) \leq 0$.

2. This suggests that $\partial_t f(t) \to 0$ as $t \to \infty$, which implies $[iA, H_t] \to 0$ and that $H_t \to H_\infty = U_\infty H_0 U_\infty^*$ with $[H_\infty, A] = 0$. 

Flow equation for operators

Natural choice: \( G_t := i[A, H_t] \). So, \( H_t \) is solution of a (quadratically) nonlinear first–order differential equation:

\[
\forall t \geq 0 : \quad \partial_t H_t = [H_t, [H_t, A]], \quad H_{t=0} := H_0.
\]


Mathematical difficulties of this idea:

1. Proof of the existence of \( H(\cdot) \) solution of the flow ?
2. Proof of the existence of \( U(\cdot) \) such that \( H_t = U_t H_0 U_t^* \) ?
3. Proof of the existence of \( H_\infty = \lim_{t \to \infty} H_t \) ?
4. Proof of the existence of \( U_\infty = \lim_{t \to \infty} U_t \) ?
5. Proof of \( H_\infty = U_\infty H_0 U_\infty^* \) with \([H_\infty, A] = 0\) ?

1. $H_0, A \in \mathcal{B}[\mathcal{H}]$ bounded operators: There is a unique, smooth solution $H(\cdot) \in C^\infty(\mathbb{R}_0^+; \mathcal{B}[\mathcal{H}])$ of

\[ \forall t \geq 0 : \quad \partial_t H_t = [H_t, [H_t, A]], \quad H_{t=0} := H_0 \tag{1} \]

and unitary operators $U(\cdot) \in C^\infty(\mathbb{R}_0^+; \mathcal{B}[\mathcal{H}])$ so that $H_t = U_t H_0 U_t^*$. 

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2. $H_0, A$ unbounded operators: The flow equation (1) has a unique, smooth local solution provided the iterated commutators of $H_0$ with $A$ define bounded operators whose norm tends to zero, as the order increases, sufficiently fast.

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2. **$H_0, A$ unbounded operators:** The flow equation (1) has a unique, smooth *local* solution provided the iterated commutators of $H_0$ with $A$ define bounded operators whose norm tends to zero, as the order increases, sufficiently fast.

3. **$H_0, A \in \mathcal{L}^2[\mathcal{H}]$ Hilbert-Schmidt operators:** The solution $H_t$ of (1) strongly converges to an operator $H_\infty$, unitarily equivalent to $H_0$, and so that $[H_\infty, A] = 0$. 
Example (Diagonalization of quadratic operators [BB’09])

- Let $\Omega_0 = \Omega_0^* \geq 0$ acting a Hilbert space $\mathcal{H}$ and unbounded.
- Let $B_0 = B_0^t \in \mathcal{L}^2(\mathcal{H})$ be a Hilbert-Schmidt operator.
- Assume that $\Omega_0^2 \geq 4B_0 \overline{B_0}$.
- Take an orthonormal basis $\{\varphi_k\}_{k=1}^{\infty}$ in $\mathcal{D}(\Omega_0) \subseteq \mathcal{H}$ and define by $a_k := a(\varphi_k)$ the bosonic annihilation operator acting on the boson Fock space $\mathcal{F}_{\Lambda^+,+}$.
- Then, for any fixed $C_0 \in \mathbb{R}$ the quadratic (unbounded) operator is

$$H_0 := \sum_{k,l} \{\Omega_0\}_{k,l} a_k^* a_l + \{B_0\}_{k,l} a_k^* a_l^* + \{\overline{B_0}\}_{k,l} a_k a_l + C_0.$$ 

($\int \text{d}k \text{d}l$ could also replace the discrete sum $\sum_{k,l}$.)

We use the flow equation $\dot{H}_t = [H_t, [H_t, A]]$ with

$$A = N := \sum_k a_k^* a_k, \quad H_{t=0} := H_0$$

to diagonalize $H_0$ under much more general assumptions as in previous works.

J.-B. Bru, V. Bach, in preparation, 78 pages ('09):
Concluding remarks

The mathematics of the quantum many-body problems is at the intersection of many different fields of mathematics:

1. Functional analysis
2. Probability theory
3. Operator algebra
4. Differential equations
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It is also related to an extremely active domain of research in physics:

1. Superconductivity
2. BEC and ultra-cold Fermi gases (to make atom lasers, atom chips, quantum simulators, etc.)

Remark: Observe that not all my personal work has been presented here.
Selected publications

32 publications. Among those are:

- Habilitation à Diriger des Recherches, Aix-Marseille II University, (100 pages, November 28, 2005).