Thin domains with a highly oscillating boundary

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We are interested in studying the problem

\[
\begin{aligned}
-\Delta w^\varepsilon + w^\varepsilon &= f^\varepsilon \quad \text{in } R^\varepsilon \\
\frac{\partial w^\varepsilon}{\partial N^\varepsilon} &= 0 \quad \text{on } \partial R^\varepsilon
\end{aligned}
\]

where \( R^\varepsilon \) is a thin domain

\[
R^\varepsilon = \{(x, y) : 0 < x < 1; 0 < y < \varepsilon g^\varepsilon(x)\}
\]

\[ g_\epsilon(x) = g(x) \]
The limit problem is

\[
\begin{aligned}
\begin{cases}
- \frac{1}{g(x)} (g(x)w_x(x))_x + w(x) = f_0(x) \quad x \in (0, 1) \\
w_x(0) = w_x(1) = 0.
\end{cases}
\end{aligned}
\]

Moreover they analyzed the asymptotic behavior of the solutions of the parabolic problem. They consider the continuity properties of the attractors $\mathcal{A}_\epsilon \subset H^1(R_\epsilon)$ when $\epsilon \geq 0$.

\[
\begin{aligned}
\begin{cases}
w_t^\epsilon - \Delta w^\epsilon + w^\epsilon = f(w^\epsilon) \quad \text{in } R_\epsilon, \ t > 0 \\
\frac{\partial w^\epsilon}{\partial N^\epsilon} = 0 \quad \text{on } \partial R_\epsilon, \ t > 0.
\end{cases}
\end{aligned}
\]
The limit equation is

\[-\frac{1}{g_i(x)}(g_i(x)w'_i(x))' + w_i(x) = f_0(x) \quad x \in J_{\Omega_i}\]

with *Kirchhoff*-type boundary conditions

\[
\sum_+ g_i(x)w'_i(x) = \sum_- g_i(x)w'_i(x).
\]

$$g_\epsilon(x) = a(x) + g(x/\epsilon^\alpha)$$

with $0 < \alpha < 1$. 
The limit problem is

\[
\begin{aligned}
&- \frac{1}{r(x)} \left( \frac{1}{s(x)} \frac{d}{dx} w(x) \right) + w(x) = f_0(x) \quad x \in (0, 1) \\
&w_x(0) = w_x(1) = 0.
\end{aligned}
\]

where

i) \( a(x) + g(x/\epsilon^\alpha) \rightharpoonup r(x), \ w - L^2(0, 1) \)

ii) \( \frac{1}{a(x)+g(x/\epsilon^\alpha)} \rightharpoonup s(x), \ w - L^2(0, 1) \).
Our setting:

• $a : (0, 1) \rightarrow \mathbb{R}, C^1$ with $0 < \alpha_0 \leq a(x) \leq \alpha_1$
• $g : \mathbb{R} \leftrightarrow \mathbb{R}, L$-periodic $C^1$, $g_0 \leq g(x) \leq g_1$ with

\[
0 < \underbrace{\alpha_0 + g_0}_{G_0} < \underbrace{\alpha_1 + g_1}_{G_1}
\]

• Our thin domain:

\[
R_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1, 0 < x_2 < \epsilon (a(x_1) + g(x_1/\epsilon))\}.
\]
CASE I: Purely periodic case.

- $a(x) = a_0$ a constant, so that $a_0 + g(x/\epsilon)$ is periodic.
- We identify the limit equation by the Multiple Scale method.
- We prove the convergence with the oscillatory test function method of Tartar.
Multiple Scale Method

\[ w^\epsilon(x_1, x_2) = w_0(x_1, \frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}) + \epsilon w_1(x_1, \frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}) + \epsilon^2 w_2(x_1, \frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}) + \ldots \]

Where \((x_1, x_2)\) are the macroscopic variables and \((\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon})\) are the microscopic variables.

Hence, if we denote \(x = x_1, y = x_1/\epsilon, z = x_2/\epsilon\).

\[
\begin{align*}
\frac{\partial}{\partial x_1} &= \partial_x + \frac{1}{\epsilon} \partial_y \\
\frac{\partial}{\partial x_2} &= \frac{1}{\epsilon} \partial_z \\
\frac{\partial^2}{\partial x_1^2} &= \partial_{xx} + \frac{2}{\epsilon} \partial_{xy} + \frac{1}{\epsilon^2} \partial_{yy} \\
\frac{\partial^2}{\partial x_2^2} &= \frac{1}{\epsilon^2} \partial_{zz}.
\end{align*}
\]
$w_i(x, y, z)$ is defined in $x \in (0, 1)$ and $(y, z) \in Y^*$, the basic cell:

$$Y^* = \{(y, z) \in \mathbb{R}^2 : 0 < y < L, 0 < z < a_0 + g(y)\},$$

We denote by $B_0, B_1$ and $B_2$ the lateral, inferior and superior boundary, respectively.
With some computations $w_0$ satisfies

$$
\begin{cases}
  -q \frac{d^2 w_0}{dx^2}(x) + w_0(x) = f(x), \quad x \in (0, 1) \\
  w'(0) = w'(1) = 0
\end{cases}
$$

where

$$
q = \frac{1}{|Y^*|} \int_{Y^*} \left\{ 1 - \frac{\partial X}{\partial y}(y, z) \right\} dydz
$$

and $X(y, z)$ is the unique solution (up to an additive constant) of

$$
\begin{cases}
  -\Delta_{y,z} X(y, z) = 0 \quad \text{in } Y^* \\
  \frac{\partial X}{\partial N}(y, g(y)) = -\frac{g'(y)}{\sqrt{1 + (g'(y))^2}} \quad \text{on } B_1 \\
  \frac{\partial X}{\partial N}(y, 0) = 0 \quad \text{on } B_2 \\
  X(0, z) = X(L, z) \quad z \in B_0.
\end{cases}
$$
Convergence result

We transform the original domain and problem with the change of variables \((x, y) \rightarrow (x, \varepsilon y)\) so

\[
\begin{aligned}
-\frac{\partial^2 u^\varepsilon}{\partial x_1^2} - \frac{1}{\varepsilon^2} \frac{\partial^2 u^\varepsilon}{\partial x_2^2} + u^\varepsilon &= f \quad \text{in } \Omega^\varepsilon \\
\frac{\partial u^\varepsilon}{\partial x_1} N_1^\varepsilon + \frac{1}{\varepsilon^2} \frac{\partial u^\varepsilon}{\partial x_2} N_2^\varepsilon &= 0 \quad \text{on } \partial \Omega^\varepsilon
\end{aligned}
\]

\[\Omega^\varepsilon = \{ (x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1, \; 0 < x_2 < a_0 + g(x_1/\varepsilon) \}\]

\[\Omega^\varepsilon \subset \Omega = (0, 1) \times (0, G_1)\]

\[f \in L^2(\Omega) \text{ with } f(x_1, x_2) = f(x_1)\].
The weak formulation of the problem: \( \forall \varphi \in H^1(\Omega^\epsilon) \)

\[
\int_{\Omega^\epsilon} \left\{ \frac{\partial u^\epsilon}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2} \frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi \right\} dx_1 dx_2 = \int_{\Omega^\epsilon} f \varphi dx_1 dx_2
\]

which, taking \( \varphi = u^\epsilon \), implies

\[
\left\| \frac{\partial u^\epsilon}{\partial x_1} \right\|^2_{L^2(\Omega^\epsilon)} + \frac{1}{\epsilon^2} \left\| \frac{\partial u^\epsilon}{\partial x_2} \right\|^2_{L^2(\Omega^\epsilon)} + \left\| u^\epsilon \right\|^2_{L^2(\Omega^\epsilon)} \leq \| f \|_{L^2} \left\| u^\epsilon \right\|_{L^2(\Omega^\epsilon)}.
\]

and this shows,

\[
\left\| u^\epsilon \right\|_{L^2(\Omega^\epsilon)}, \left\| \frac{\partial u^\epsilon}{\partial x_1} \right\|_{L^2(\Omega^\epsilon)}, \ y \frac{1}{\epsilon} \left\| \frac{\partial u^\epsilon}{\partial x_2} \right\|_{L^2(\Omega^\epsilon)} \leq M \ \forall \epsilon > 0.
\]
Hence, via subsequences we have

\[ \tilde{u}^\varepsilon \rightharpoonup u^* \quad w - L^2(\Omega) \]

\[ \frac{\partial u^\varepsilon}{\partial x_1} \rightharpoonup \xi^* \quad w - L^2(\Omega) \quad y \]

\[ \frac{\partial u^\varepsilon}{\partial x_2} \rightarrow 0 \quad s - L^2(\Omega) \]

where \( \tilde{\cdot} \) is the extension by zero from \( \Omega_\varepsilon \) to \( \Omega \).
Let \( \chi \) denote the periodic extension in the \( y \) variable of the characteristic function of the basic cell \( Y^* \). Therefore,

\[
\chi^\epsilon(x_1, x_2) = \chi\left(\frac{x_1 - \epsilon k L}{\epsilon}, x_2\right) = \chi\left(\frac{x_1}{\epsilon}, x_2\right)
\]

with \( k \in \mathbb{N} \) such that \((y, z) = (\frac{x_1 - \epsilon k L}{\epsilon}, x_2) \in Y^* \). Hence,

\[
\chi^\epsilon(x_1, x_2) \rightharpoonup \theta(x_2) := \frac{1}{L} \int_0^L \chi(s, x_2) ds \quad w^* - L^\infty(\Omega)
\]

as \( \epsilon \to 0 \), for all \( x_2 \in (0, G_1) \). Hence, by the Dominated Convergence Theorem

\[
\chi^\epsilon \rightharpoonup \theta \quad w^* - L^\infty(\Omega).
\]
Hence, we can pass to the limit in the weak formulation of the problem and we obtain,

\[ \int_{\Omega} \left\{ \xi^* \frac{\partial \varphi}{\partial x_1} + u^* \varphi \right\} dx_1 dx_2 = \int_{\Omega} \theta f \varphi dx_1 dx_2 \]

for all \( \varphi \in H^1(0, 1) \).

What is the relation among \( u^* \), \( \xi^* \) and \( \theta \)?
We need the following ingredients:

- An extension operator
- Oscillatory test functions method of Tartar
Extension operator

Let $\Omega$ and $\Omega_\varepsilon$ defined as

$$
\Omega = \{(x_1, x_2) \in \mathbb{R}^2 | 0 < x_1 < 1, \, y \, 0 < x_2 < G_1\} \\
\Omega_\varepsilon = \{(x_1, x_2) \in \mathbb{R}^2 | 0 < x_1 < 1 \, e\, 0 < x_2 < G_\varepsilon(x_1)\}
$$

with $0 < G_0 \leq G_\varepsilon(x_1) \leq G_1$. Then, we can construct an extension operator

$$
P_\varepsilon \in \mathcal{L}(L^p(\Omega_\varepsilon), L^p(\Omega)) \cap \mathcal{L}(W^{1,p}(\Omega_\varepsilon), W^{1,p}(\Omega))
$$

satisfying

$$
\|P_\varepsilon \varphi\|_{L^p(\Omega)} \leq K \|\varphi\|_{L^p(\Omega_\varepsilon)}
$$

$$
\left\| \frac{\partial P_\varepsilon \varphi}{\partial x_1} \right\|_{L^p(\Omega)} \leq K \left\{ \left\| \frac{\partial \varphi}{\partial x_1} \right\|_{L^p(\Omega_\varepsilon)} + \eta(\varepsilon) \left\| \frac{\partial \varphi}{\partial x_2} \right\|_{L^p(\Omega_\varepsilon)} \right\}
$$

$$
\left\| \frac{\partial P_\varepsilon \varphi}{\partial x_2} \right\|_{L^p(\Omega)} \leq K \left\| \frac{\partial \varphi}{\partial x_2} \right\|_{L^p(\Omega_\varepsilon)}
$$

for all $\varphi \in W^{1,p}(\Omega_\varepsilon)$ with $1 \leq p \leq \infty$ and

$$
\eta(\varepsilon) = \sup_{x \in I}\{|G'_\varepsilon(x)|\}.
$$
With this extension operator,

\[ \| P_\epsilon u^\epsilon \|_{L^2(\Omega)}, \| \frac{\partial P_\epsilon u^\epsilon}{\partial x_1} \|_{L^2(\Omega)}, \frac{1}{\epsilon} \| \frac{\partial P_\epsilon u^\epsilon}{\partial x_2} \|_{L^2(\Omega)} \leq \tilde{M} \]

where \( \tilde{M} > 0 \) independent of \( \epsilon > 0 \).

We can take a subsequence \( P_\epsilon u^\epsilon \) so that

- \( P_\epsilon u^\epsilon \rightharpoonup u_0 \) \( w - H^1(\Omega) \) and \( s - L^2(\Omega) \)
- \( \frac{\partial P_\epsilon u^\epsilon}{\partial x_2} \to 0 \) \( s - L^2(\Omega) \).

Hence, \( u_0(x_1, x_2) = u_0(x_1) \), that is

\[ \frac{\partial u_0}{\partial x_2}(x_1, x_2) = 0 \text{ a.e. } \Omega. \]
Moreover, since $\widetilde{u}^\epsilon = \chi^\epsilon P_\epsilon u^\epsilon$ a.e. $\Omega$, passing to the limit, we get

$$u^*(x_1, x_2) = \theta(x_2) u_0(x_1) \quad \text{a.e. } \Omega.$$
Hence, the weak formulation

\[
\int_{\Omega} \left\{ \xi^* \frac{\partial \varphi}{\partial x_1} + u^* \varphi \right\} dx_1 dx_2 = \int_{\Omega} \theta f \varphi dx_1 dx_2, \quad \forall \varphi \in H^1(0, 1)
\]

can be written as

\[
\int_0^1 \left\{ \left( \int_0^{G_1} \xi^*(x_1, x_2) dx_2 \right) \frac{\partial \varphi}{\partial x_1} + \left( \int_0^{G_1} \theta(x_2) dx_2 \right) u_0(x_1) \varphi(x_1) \right\} dx_1
\]

\[
= \int_0^1 \left( \int_0^{G_1} \theta(x_2) dx_2 \right) f(x_1) \varphi(x_1) dx_1, \quad \forall \varphi \in H^1(0, 1)
\]

But,

\[
\int_0^{G_1} \theta(x_2) dx_2 = \frac{|Y^*|}{L}
\]
Which shows that

\[ \int_0^1 \left\{ \left( \int_0^{G_1} \xi^*(x_1, x_2) \, dx_2 \right) \frac{\partial \varphi}{\partial x_1} + \frac{|Y^*|}{L} u_0(x_1) \varphi(x_1) \right\} \, dx_1 \]

\[ = \int_0^1 \frac{|Y^*|}{L} f(x_1) \varphi(x_1) \, dx_1, \quad \forall \varphi \in H^1(0, 1) \]
Oscillatory test functions method of Tartar

In order to identify the function

$$x_1 \rightarrow \int_0^{G_1} \xi^*(x_1, x_2) dx_2$$

we use the oscillatory test functions method. We obtain that,

$$\int_0^{G_1} \xi^*(x_1, x_2) dx_2 = q \frac{|Y^*|}{L} \frac{\partial u_0}{\partial x_1}$$

where

$$q = \frac{1}{|Y^*|} \int_{Y^*} \left( 1 - \frac{\partial X}{\partial y_1}(y_1, y_2) \right) dy_1 dy_2$$
Where,

\[-\Delta_{y,z} X(y, z) = 0 \quad \text{in } Y^*\]

\[\frac{\partial X}{\partial N}(y, g(y)) = -\frac{g'(y)}{\sqrt{1 + (g'(y))^2}} \quad \text{on } B_1\]

\[\frac{\partial X}{\partial N}(y, 0) = 0 \quad \text{on } B_2\]

\[X(0, z) = X(L, z) \quad z \in B_0.\]
Which implies that the weak formulation of the limit problem is

\[
\int_0^1 \left\{ q \frac{|Y^*|}{L} \frac{\partial u_0}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{|Y^*|}{L} u_0(x_1) \varphi(x_1) \right\} dx_1
\]

\[
= \int_0^1 \frac{|Y^*|}{L} f(x_1) \varphi(x_1) dx_1, \quad \forall \varphi \in H^1(0, 1)
\]

or equivalently,

\[
\int_0^1 (q \frac{\partial u_0}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + u_0 \varphi) dx_1 = \int_0^1 f \varphi dx_1, \quad \forall \varphi \in H^1(0, 1)
\]
Hence, the limit problem is:

\[
\begin{align*}
- qu_0''(x) + u_0(x) &= f(x), & x \in (0, 1) \\
 u_0'(0) &= u_0'(1) = 0
\end{align*}
\]
CASE II: Piecewise periodic case.

\[ \Omega_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1, 0 < x_2 < a(x_1) + g(x_1/\epsilon)\}. \]

where \( a(x) = a_i^j \) for \( x \in I_i \) and \( (0, 1) = I_1 \cup \cdots \cup I_K \) and 
\( \alpha_0 \leq a_0^j \leq \alpha_1. \)
The extension by zero still works.

\[ \tilde{u}^e \rightarrow u^* \quad w - L^2(\Omega) \]

\[ \frac{\partial u^e}{\partial x_1} \rightarrow \xi^* \quad w - L^2(\Omega) \quad y \]

\[ \frac{\partial u^e}{\partial x_2} \rightarrow 0 \quad s - L^2(\Omega) \]
And with similar arguments we may find

$$
\sum_{i=1}^{K} \int_{l_i} q_i \left( \frac{|Y_i^*|}{L} \frac{\partial u_0}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{|Y_i^*|}{L} u_0(x_1) \varphi(x_1) \right) dx_1
$$

$$
= \sum_{i=1}^{K} \int_{l_i} \frac{|Y_i^*|}{L} f(x_1) \varphi(x_1) dx_1, \quad \forall \varphi \in H^1(0, 1)
$$
CASE III: general case.

$$\Omega_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1, \ 0 < x_2 < a(x_1) + g(x_1/\epsilon)\}.$$
The previous case suggests that the limit should be:

\[
\int_0^1 q(x_1) \left\{ \frac{|Y^*(x_1)|}{L} \frac{\partial u_0}{\partial x_1} \frac{\partial \phi}{\partial x_1} + \frac{|Y^*(x)|}{L} u_0(x_1) \phi(x_1) \right\} \, dx_1
\]

\[
= \int_0^1 \frac{|Y^*(x_1)|}{L} f(x_1) \phi(x_1) \, dx_1, \quad \forall \phi \in H^1(0, 1)
\]

Where

\[
Y^*(x_1) = \{(y, z) \in \mathbb{R}^2 : 0 < y < L, 0 < z < a(x_1) + g(y)\},
\]

and

\[
q(x_1) = \frac{1}{|Y^*(x_1)|} \int_{Y^*(x_1)} \left( 1 - \frac{\partial X}{\partial y_1}(x_1, y, z) \right) \, dy\,dz
\]
Where,

\[
\begin{align*}
-\Delta_{y,z} X(x_1, y, z) &= 0 \quad \text{in } Y^*(x_1) \\
\frac{\partial X}{\partial N}(x_1, y, g(y)) &= -\frac{g'(y)}{\sqrt{1 + (g'(y))^2}} \quad \text{on } B_1(x_1) \\
\frac{\partial X}{\partial N}(x_1, y, 0) &= 0 \quad \text{on } B_2(x_1) \\
X(x_1, 0, z) &= X(x_1, L, z) \quad z \in B_0(x_1).
\end{align*}
\]
\[ \Omega^\delta_{\epsilon} = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1, 0 < x_2 < a_\delta(x_1) + g(x_1/\epsilon)\}. \]

\[ \Omega_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1, 0 < x_2 < a(x_1) + g(x_1/\epsilon)\}. \]

where \( a_\delta(x_1) \) is a piecewise constant function satisfying

\[ \|a_\delta - a\|_{L^\infty(0,1)} \leq \delta. \]
\[
\downarrow (\epsilon \to 0) \downarrow
\]

\[(\text{Equation})_{\delta} \xrightarrow{\delta \to 0} (\text{Equation})_{0}\]

\[(\text{Eq})_{\delta} : \quad \int_{0}^{1} (q_{\delta} | Y_{\delta}^{*} u_{0}' \varphi' + | Y_{\delta}^{*} u_{0} \varphi) \, dx_{1} = \int_{0}^{1} | Y_{\delta}^{*} f \varphi \, dx_{1}\]

and \quad q_{\delta} = q_{\delta}(x), \quad Y_{\delta}^{*} = Y_{\delta}^{*}(x)\]
\[
\downarrow (\epsilon \to 0) \downarrow \quad \text{(Equation)}_\delta \quad \delta \to 0 \quad \downarrow \quad \text{(Equation)}_0
\]

\[
\begin{align*}
\text{(Eq)}_\delta: \quad & \int_0^1 (q_\delta |Y_{\delta}^* u_0' \varphi' + Y_{\delta}^* u_0 \varphi) \, dx_1 = \int_0^1 |Y_{\delta}^* f \varphi \, dx_1 \\
\text{and} \quad & q_\delta = q_\delta(x), \quad Y_{\delta}^* = Y_{\delta}^*(x)
\end{align*}
\]
The convergence \((Eq)\delta \xrightarrow{\delta \to 0} (Eq)\) follows from:

\[
Y_\delta^*(x) = \{ (y, z) \in \mathbb{R}^2 : 0 < y < L, 0 < z < a_\delta(x) + g(y) \},
\]

\[
q_\delta(x) = \frac{1}{|Y_\delta^*(x)|} \int_{Y_\delta^*(x)} \left( 1 - \frac{\partial X_\delta}{\partial y_1}(x, y_1, y_2) \right) dy_1 dy_2
\]

and \(X_\delta(x, \cdot, \cdot)\) satisfies

\[
\begin{cases}
-\Delta_{y,z} X_\delta(x, y, z) = 0 & \text{in } Y_\delta^*(x) \\
BC_\delta(X_\delta) = 0
\end{cases}
\]
We are going to show a continuous dependence result on the function \( a(x) \) for the solutions in the thin domains:

\[
\Omega_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1, 0 < x_2 < a(x_1) + g(x_1/\epsilon)\}.
\]

\[
\hat{\Omega}_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1, 0 < x_2 < \hat{a}(x_1) + g(x_1/\epsilon)\}.
\]

with \( \alpha_0 \leq a(x), \hat{a}(x) \leq \beta_0 \).
Denote by $u_\epsilon$ and $\hat{u}_\epsilon$ the solutions of

$$
\begin{aligned}
&\left\{\begin{array}{ll}
-\frac{\partial^2 u_\epsilon}{\partial x_1^2} - \frac{1}{\epsilon^2} \frac{\partial^2 u_\epsilon}{\partial x_2^2} + u_\epsilon = f_\epsilon & \text{in } \Omega_\epsilon \\
\frac{\partial u_\epsilon}{\partial x_1} N_1^\epsilon + \frac{1}{\epsilon^2} \frac{\partial u_\epsilon}{\partial x_2} N_2^\epsilon = 0 & \text{on } \partial \Omega_\epsilon
\end{array}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\left\{\begin{array}{ll}
-\frac{\partial^2 \hat{u}_\epsilon}{\partial x_1^2} - \frac{1}{\epsilon^2} \frac{\partial^2 \hat{u}_\epsilon}{\partial x_2^2} + \hat{u}_\epsilon = f_\epsilon & \text{in } \hat{\Omega}_\epsilon \\
\frac{\partial \hat{u}_\epsilon}{\partial x_1} \hat{N}_1^\epsilon + \frac{1}{\epsilon^2} \frac{\partial \hat{u}_\epsilon}{\partial x_2} \hat{N}_2^\epsilon = 0 & \text{on } \partial \hat{\Omega}_\epsilon
\end{array}\right.
\end{aligned}
$$

with $f_\epsilon \in L^2(\mathbb{R}^2)$. 
Theorem

There exists a function $\rho(\delta) \to 0$ as $\delta \to 0$ such that

$$\| u_\epsilon - \hat{u}_\epsilon \|^2_{H^1_\epsilon(\Omega_\epsilon \cap \hat{\Omega}_\epsilon)} + \| u_\epsilon \|^2_{H^1_\epsilon(\Omega_\epsilon \setminus \hat{\Omega}_\epsilon)} + \| \hat{u}_\epsilon \|^2_{H^1_\epsilon(\hat{\Omega}_\epsilon \setminus \Omega_\epsilon)} \leq \rho(\delta)$$

uniformly for all

- $\epsilon \in (0, \epsilon_0)$
- piecwise $C^1$ functions $a, \hat{a}$ with $\| a - \hat{a} \|_{L^\infty(0,1)} \leq \delta$
- $\alpha_0 \leq a(x), \hat{a}(x) \leq \alpha_1$
- $f_\epsilon \in L^2(\mathbb{R}^2)$, $\| f_\epsilon \|_{L^2(\mathbb{R}^2)} \leq 1$

Remark: The space, $H^1_\epsilon(U) = H^1(U)$ with the norm

$$\| u \|^2_{H^1_\epsilon(U)} = \| u \|^2_{L^2(U)} + \| u_{x_1} \|^2_{L^2(U)} + \frac{1}{\epsilon^2} \| u_{x_2} \|^2_{L^2(U)}$$