Thin domains with a highly oscillating boundary

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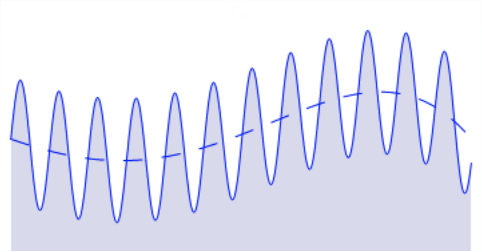
We are interested in studying the problem

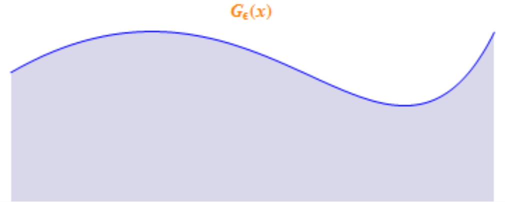
$$\begin{cases} -\Delta w^\epsilon + w^\epsilon = f_\epsilon & R_\epsilon \\ \dfrac{\partial w^\epsilon}{\partial N^\epsilon} = 0 & \partial R_\epsilon \end{cases}$$

where R_ϵ is a thin domain

$$R_\epsilon = \{(x, y) : 0 < x < 1; 0 < y < \epsilon g_\epsilon(x)\}$$



J. K. Hale and G. Raugel *Reaction-diffusion equation on thin domains*, J. Math. Pures and Appl. (9) 71, no. 1, 33-95 (1992).



$$g_\epsilon(x) = g(x)$$

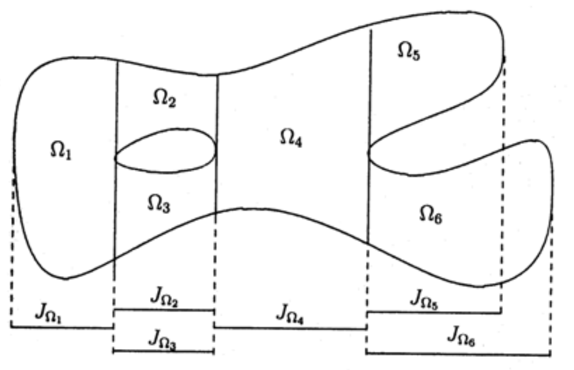
The limit problem is

$$\begin{cases} -\dfrac{1}{g(x)}(g(x)w_x(x))_x + w(x) = f_0(x) \quad x \in (0,1) \\ w_x(0) = w_x(1) = 0. \end{cases}$$

Moreover they analyzed the asymptotic behavior of the solutions of the parabolic problem. They consider the continuity properties of the attractors $\mathcal{A}_\epsilon \subset H^1(R_\epsilon)$ when $\epsilon \geq 0$.

$$\begin{cases} w_t^\epsilon - \Delta w^\epsilon + w^\epsilon = f(w^\epsilon) \quad \text{in } R_\epsilon,\ t > 0 \\ \dfrac{\partial w^\epsilon}{\partial N^\epsilon} = 0 \quad \text{on } \partial R_\epsilon,\ t > 0. \end{cases}$$

Prizzi and Rybakowski *The effect of domain squeezing upon the dynamics of reaction-diffusion equations*, J. of Diff. Equation 173, no. 2, 271-320 (2001).



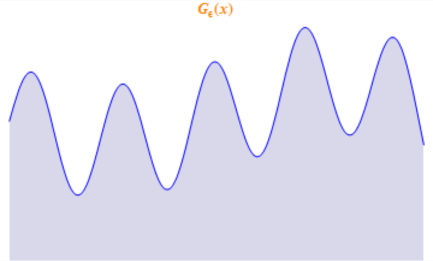
The limit equation is

$$-\frac{1}{g_i(x)}(g_i(x)w_i'(x))' + w_i(x) = f_0(x) \quad x \in J_{\Omega_i}$$

with *Kirchhoff*-type boundary conditions

$$\sum_{+} g_i(x)w_i'(x) = \sum_{-} g_i(x)w_i'(x).$$

J. A. *Spectral properties of Schrödinger operators under perturbations of the domain*, Ph.D. Thesis - 1991.



$$g_\epsilon(x) = a(x) + g(x/\epsilon^\alpha)$$

with $0 < \alpha < 1$.

The limit problem is

$$\begin{cases} -\dfrac{1}{r(x)}\left(\dfrac{1}{s(x)}w_x(x)\right)_x + w(x) = f_0(x) \quad x \in (0,1) \\ w_x(0) = w_x(1) = 0. \end{cases}$$

where

i) $a(x) + g(x/\epsilon^\alpha) \rightharpoonup r(x)$, $w - L^2(0,1)$

ii) $\frac{1}{a(x)+g(x/\epsilon^\alpha)} \rightharpoonup s(x)$, $w - L^2(0,1)$.

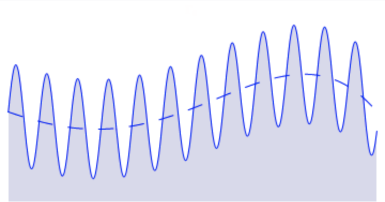
Our setting:

- $a : (0,1) \to \mathbb{R}$, $\mathcal{C}^1$ with $0 < \alpha_0 \leq a(x) \leq \alpha_1$
- $g : \mathbb{R} \mapsto \mathbb{R}$, L-periodic $\mathcal{C}^1$, $g_0 \leq g(x) \leq g_1$ with

$$0 < \underbrace{\alpha_0 + g_0}_{G_0} < \underbrace{\alpha_1 + g_1}_{G_1}$$

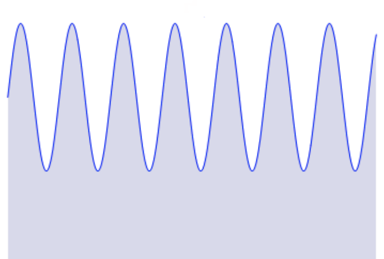
- Our thin domain:

$$R_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1,\ 0 < x_2 < \epsilon(a(x_1) + g(x_1/\epsilon))\}.$$



CASE I: Purely periodic case.

- $a(x) = a_0$ a constant, so that $a_0 + g(x/\epsilon)$ is periodic.
- We identify the limit equation by the Multiple Scale method.
- We prove the convergence with the oscillatory test function method of Tartar.



Multiple Scale Method

$$w^\epsilon(x_1,x_2) = w_0(x_1,\frac{x_1}{\epsilon},\frac{x_2}{\epsilon}) + \epsilon w_1(x_1,\frac{x_1}{\epsilon},\frac{x_2}{\epsilon}) + \epsilon^2 w_2(x_1,\frac{x_1}{\epsilon},\frac{x_2}{\epsilon}) + \dots$$

Where (x_1,x_2) are the macroscopic variables and $(\frac{x_1}{\epsilon},\frac{x_2}{\epsilon})$ are the microscopic variables.
Hence, if we denote $x = x_1$, $y = x_1/\epsilon$, $z = x_2/\epsilon$.

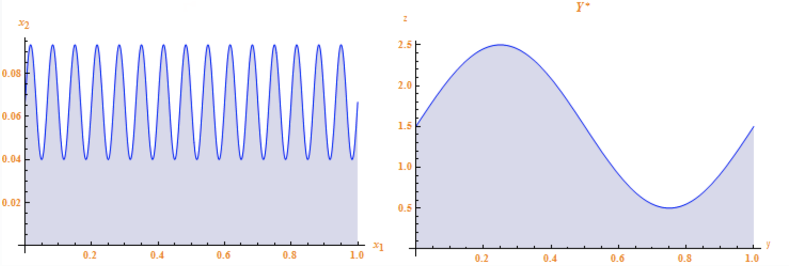
$$\frac{\partial}{\partial x_1} = \partial_x + \frac{1}{\epsilon}\partial_y \qquad \frac{\partial}{\partial x_2} = \frac{1}{\epsilon}\partial_z$$

$$\frac{\partial^2}{\partial {x_1}^2} = \partial_{xx} + \frac{2}{\epsilon}\partial_{xy} + \frac{1}{\epsilon^2}\partial_{yy} \qquad \frac{\partial^2}{\partial {x_2}^2} = \frac{1}{\epsilon^2}\partial_{zz}.$$

$w_i(x,y,z)$ is defined in $x \in (0,1)$ and $(y,z) \in Y^*$, the basic cell:

$$Y^* = \{(y,z) \in \mathbb{R}^2 \ : \ 0 < y < L, \, 0 < z < a_0 + g(y)\},$$

We denote by B_0,B_1 y B_2 the lateral, inferior and superior boundary, respectively.



With some computations w_0 satisfies

$$\begin{cases} -q\dfrac{d^2 w_0}{dx^2}(x) + w_0(x) = f(x), \quad x \in (0,1) \\ w_0'(0) = w_0'(1) = 0 \end{cases}$$

where

$$q = \frac{1}{|Y^*|}\int_{Y^*}\left\{1 - \frac{\partial X}{\partial y}(y,z)\right\}dydz$$

and $X(y,z)$ is the unique solution (up to an additive constant) of

$$\begin{cases} -\Delta_{y,z}X(y,z) = 0 \quad \text{in } Y^* \\ \dfrac{\partial X}{\partial N}(y,g(y)) = -\dfrac{g'(y)}{\sqrt{1+(g'(y))^2}} \quad \text{on } B_1 \\ \dfrac{\partial X}{\partial N}(y,0) = 0 \quad \text{on } B_2 \\ X(0,z) = X(L,z) \quad z \in B_0. \end{cases}$$

Convergence result

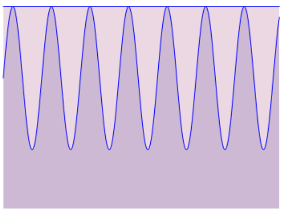
We transform the original domain and problem with the change of variables $(x, y) \to (x, \epsilon y)$ so

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f \quad \text{in } \Omega^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial x_1} N_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2} N_2^\epsilon = 0 \quad \text{on } \partial\Omega^\epsilon \end{cases}$$

$$\Omega^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1,\ 0 < x_2 < a_0 + g(x_1/\epsilon)\}$$

$$\Omega^\epsilon \subset \Omega = (0,1) \times (0, G_1)$$

$$f \in L^2(\Omega) \text{ with } f(x_1, x_2) = f(x_1).$$



The weak formulation of the problem: $\forall \varphi \in H^1(\Omega^\epsilon)$

$$\int_{\Omega^\epsilon} \Big\{ \frac{\partial u^\epsilon}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2} \frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi \Big\} dx_1 dx_2 = \int_{\Omega^\epsilon} f \varphi dx_1 dx_2$$

which, taking $\varphi = u_\epsilon$, implies

$$\Big\| \frac{\partial u^\epsilon}{\partial x_1} \Big\|^2_{L^2(\Omega_\epsilon)} + \frac{1}{\epsilon^2} \Big\| \frac{\partial u^\epsilon}{\partial x_2} \Big\|^2_{L^2(\Omega_\epsilon)} + \|u^\epsilon\|^2_{L^2(\Omega_\epsilon)} \leq \|f\|_{L^2} \|u^\epsilon\|_{L^2(\Omega_\epsilon)}.$$

and this shows,

$$\|u^\epsilon\|_{L^2(\Omega_\epsilon)}, \Big\| \frac{\partial u^\epsilon}{\partial x_1} \Big\|_{L^2(\Omega_\epsilon)} \text{ y } \frac{1}{\epsilon} \Big\| \frac{\partial u^\epsilon}{\partial x_2} \Big\|_{L^2(\Omega_\epsilon)} \leq M \; \forall \epsilon > 0.$$

Hence, via subsequences we have

$$\widetilde{u^\epsilon} \rightharpoonup u^* \quad w - L^2(\Omega)$$
$$\widetilde{\frac{\partial u^\epsilon}{\partial x_1}} \rightharpoonup \xi^* \quad w - L^2(\Omega) \text{ y}$$
$$\widetilde{\frac{\partial u^\epsilon}{\partial x_2}} \to 0 \quad s - L^2(\Omega)$$

where $\widetilde{\cdot}$ is the extension by zero from Ω_ϵ to Ω.

Let χ denote the periodic extension in the y variable of the characteristic function of the basic cell Y^*. Therefore,

$$\chi^\epsilon(x_1, x_2) = \chi(\frac{x_1 - \epsilon kL}{\epsilon}, x_2) = \chi(\frac{x_1}{\epsilon}, x_2)$$

with $k \in \mathbb{N}$ such that $(y, z) = (\frac{x_1 - \epsilon kL}{\epsilon}, x_2) \in Y^*$. Hence,

$$\chi^\epsilon(x_1, x_2) \rightharpoonup \theta(x_2) := \frac{1}{L}\int_0^L \chi(s, x_2)ds \quad w^* - L^\infty(I)$$

as $\epsilon \to 0$, for all $x_2 \in (0, G_1)$. Hence, by the Dominated Convergence Theorem

$$\chi^\epsilon \rightharpoonup \theta \quad w^* - L^\infty(\Omega).$$

Hence, we can pass to the limit in the weak formulation of the problem and we obtain,

$$\int_\Omega \left\{\xi^* \frac{\partial \varphi}{\partial x_1} + u^* \varphi\right\} dx_1 dx_2 = \int_\Omega \theta f \varphi dx_1 dx_2$$

for all $\varphi \in H^1(0,1)$.

What is the relation among u^*, ξ^* and θ?

We need the following ingredients:

- An extension operator
- Oscillatory test functions method of Tartar

Extension operator

Let Ω and Ω_ϵ defined as

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1, \text{ y } 0 < x_2 < G_1\}$$
$$\Omega_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1 \text{ e } 0 < x_2 < G_\epsilon(x_1)\}$$

with $0 < G_0 \leq G_\epsilon(x_1) \leq G_1$. Then, we can construct an extension operator

$$P_\epsilon \in \mathcal{L}(L^p(\Omega_\epsilon), L^p(\Omega)) \cap \mathcal{L}(W^{1,p}(\Omega_\epsilon), W^{1,p}(\Omega))$$

satisfying

$$\|P_\epsilon \varphi\|_{L^p(\Omega)} \leq K \|\varphi\|_{L^p(\Omega_\epsilon)}$$
$$\left\| \frac{\partial P_\epsilon \varphi}{\partial x_1} \right\|_{L^p(\Omega)} \leq K \left\{ \left\| \frac{\partial \varphi}{\partial x_1} \right\|_{L^p(\Omega_\epsilon)} + \eta(\epsilon) \left\| \frac{\partial \varphi}{\partial x_2} \right\|_{L^p(\Omega_\epsilon)} \right\}$$
$$\left\| \frac{\partial P_\epsilon \varphi}{\partial x_2} \right\|_{L^p(\Omega)} \leq K \left\| \frac{\partial \varphi}{\partial x_2} \right\|_{L^p(\Omega_\epsilon)}$$

for all $\varphi \in W^{1,p}(\Omega_\epsilon)$ with $1 \leq p \leq \infty$ and

$$\eta(\epsilon) = \sup_{x \in I} \{|G'_\epsilon(x)|\}.$$

With this extension operator,

$$\|P_\epsilon u^\epsilon\|_{L^2(\Omega)}, \left\|\frac{\partial P_\epsilon u^\epsilon}{\partial x_1}\right\|_{L^2(\Omega)}, \frac{1}{\epsilon}\left\|\frac{\partial P_\epsilon u^\epsilon}{\partial x_2}\right\|_{L^2(\Omega)} \leq \tilde{M}$$

where $\tilde{M} > 0$ independent of $\epsilon > 0$.

We can take a subsequence $P_\epsilon u^\epsilon$ so that

- $P_\epsilon u^\epsilon \rightharpoonup u_0 \quad w - H^1(\Omega)$ and $s - L^2(\Omega)$
- $\dfrac{\partial P_\epsilon u^\epsilon}{\partial x_2} \to 0 \quad s - L^2(\Omega).$

Hence, $u_0(x_1, x_2) = u_0(x_1)$, that is

$$\frac{\partial u_0}{\partial x_2}(x_1, x_2) = 0 \text{ a.e. } \Omega.$$

Moreover, since $\widetilde{u^\epsilon} = \chi^\epsilon P_\epsilon u^\epsilon$ a.e. Ω, passing to the limit, we get

$$u^*(x_1, x_2) = \theta(x_2)\, u_0(x_1) \quad \text{a.e. } \Omega.$$

Hence, the weak formulation

$$\int_\Omega \left\{\xi^* \frac{\partial \varphi}{\partial x_1} + u^* \varphi\right\} dx_1 dx_2 = \int_\Omega \theta f \varphi dx_1 dx_2, \;\; \forall \varphi \in H^1(0,1)$$

can be written as

$$\int_0^1 \left\{ \left(\int_0^{G_1} \xi^*(x_1, x_2) dx_2 \right) \frac{\partial \varphi}{\partial x_1} + \left(\int_0^{G_1} \theta(x_2) dx_2 \right) u_0(x_1) \varphi(x_1) \right\} dx_1$$

$$= \int_0^1 \left(\int_0^{G_1} \theta(x_2) dx_2 \right) f(x_1) \varphi(x_1) dx_1, \;\; \forall \varphi \in H^1(0,1)$$

But,

$$\int_0^{G_1} \theta(x_2) dx_2 = \frac{|Y^*|}{L}$$

Which shows that

$$\int_0^1 \left\{ \left(\int_0^{G_1} \xi^*(x_1, x_2) dx_2 \right) \frac{\partial \varphi}{\partial x_1} + \frac{|Y^*|}{L} u_0(x_1) \varphi(x_1) \right\} dx_1$$

$$= \int_0^1 \frac{|Y^*|}{L} f(x_1) \varphi(x_1) dx_1, \quad \forall \varphi \in H^1(0,1)$$

Oscillatory test functions method of Tartar

In order to identify the function

$$x_1 \to \int_0^{G_1} \xi^*(x_1, x_2) dx_2$$

we use the oscillatory test functions method.
We obtain that,

$$\int_0^{G_1} \xi^*(x_1, x_2) dx_2 = q \frac{|Y^*|}{L} \frac{\partial u_0}{\partial x_1}$$

where

$$q = \frac{1}{|Y^*|} \int_{Y^*} \left(1 - \frac{\partial X}{\partial y_1}(y_1, y_2) \right) dy_1 dy_2$$

Where,

$$\begin{cases} -\Delta_{y,z} X(y,z) = 0 \quad \text{in } Y^* \\ \dfrac{\partial X}{\partial N}(y, g(y)) = -\dfrac{g'(y)}{\sqrt{1+(g'(y))^2}} \quad \text{on } B_1 \\ \dfrac{\partial X}{\partial N}(y, 0) = 0 \quad \text{on } B_2 \\ X(0,z) = X(L,z) \quad z \in B_0. \end{cases}$$

Which implies that the weak formulation of the limit problem is

$$\int_0^1 q\frac{|Y^*|}{L}\frac{\partial u_0}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{|Y^*|}{L}u_0(x_1)\varphi(x_1)\Big\}dx_1$$

$$= \int_0^1 \frac{|Y^*|}{L}f(x_1)\varphi(x_1)dx_1, \ \ \forall\varphi \in H^1(0,1)$$

or equivalenly,

$$\int_0^1 (q\frac{\partial u_0}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + u_0\varphi)dx_1 = \int_0^1 f\varphi dx_1, \ \ \forall\varphi \in H^1(0,1)$$

Hence, the limit problem is:

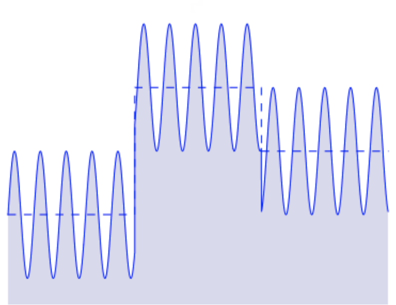
Homogenized problem

$$\begin{cases} -qu_0''(x) + u_0(x) = f(x), \quad x \in (0,1) \\ u_0'(0) = u_0'(1) = 0 \end{cases}$$

CASE II: Piecewise periodic case.

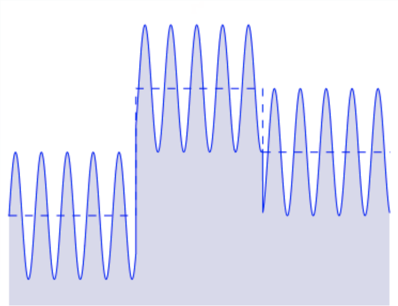
$$\Omega_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1, 0 < x_2 < a(x_1) + g(x_1/\epsilon)\}.$$

where $a(x) = a_0^i$ for $x \in I_i$ and $(0,1) = I_1 \cup \cdots \cup I_K$ and $\alpha_0 \le a_0^i \le \alpha_1$.



- The extension by zero still works.

$$\widetilde{u^\epsilon} \rightharpoonup u^* \quad w - L^2(\Omega)$$
$$\widetilde{\frac{\partial u^\epsilon}{\partial x_1}} \rightharpoonup \xi^* \quad w - L^2(\Omega) \text{ y}$$
$$\widetilde{\frac{\partial u^\epsilon}{\partial x_2}} \to 0 \quad s - L^2(\Omega)$$



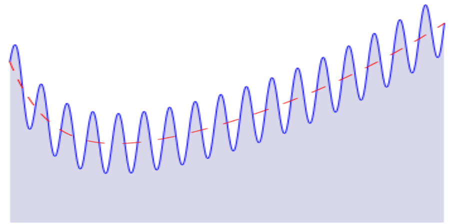
And with similar arguments we may find

$$\sum_{i=1}^{K}\int_{I_i} q_i \frac{|Y_i^*|}{L}\frac{\partial u_0}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{|Y_i^*|}{L} u_0(x_1)\varphi(x_1)\Big\} dx_1$$

$$= \sum_{i=1}^{K}\int_{I_i} \frac{|Y_i^*|}{L} f(x_1)\varphi(x_1) dx_1, \;\; \forall \varphi \in H^1(0,1)$$

CASE III: general case.

$$\Omega_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1,\ 0 < x_2 < a(x_1) + g(x_1/\epsilon)\}.$$



The previous case suggests that the limit should be:

$$\int_0^1 q(x_1)\frac{|Y^*(x_1)|}{L}\frac{\partial u_0}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{|Y^*(x)|}{L}u_0(x_1)\varphi(x_1)\Big\}dx_1$$

$$= \int_0^1 \frac{|Y^*(x_1)|}{L}f(x_1)\varphi(x_1)dx_1, \ \ \forall\varphi \in H^1(0,1)$$

Where

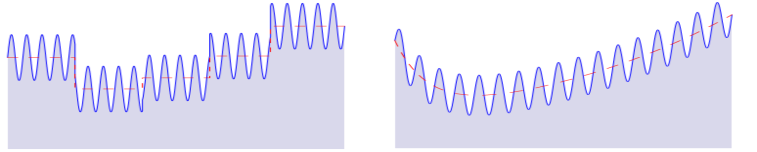
$$Y^*(x_1) = \{(y,z) \in \mathbb{R}^2 \ : \ 0 < y < L, 0 < z < a(x_1) + g(y)\},$$

and

$$q(x_1) = \frac{1}{|Y^*(x_1)|}\int_{Y^*(x_1)}\left(1 - \frac{\partial X}{\partial y_1}(x_1,y,z)\right)dydz$$

Where,

$$
\begin{cases}
-\Delta_{y,z} X(x_1, y, z) = 0 \quad \text{in } Y^*(x_1) \\
\dfrac{\partial X}{\partial N}(x_1, y, g(y)) = -\dfrac{g'(y)}{\sqrt{1 + (g'(y))^2}} \quad \text{on } B_1(x_1) \\
\dfrac{\partial X}{\partial N}(x_1, y, 0) = 0 \quad \text{on } B_2(x_1) \\
X(x_1, 0, z) = X(x_1, L, z) \quad z \in B_0(x_1).
\end{cases}
$$

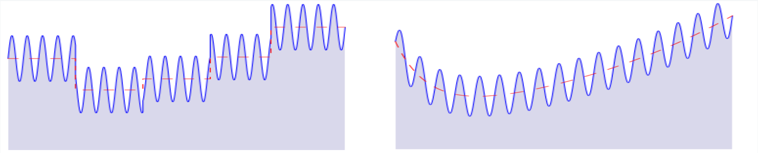


Ω_ϵ^δ $\qquad\qquad$ Ω_ϵ

$$\Omega_\epsilon^\delta = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1,\ 0 < x_2 < a_\delta(x_1) + g(x_1/\epsilon)\}.$$
$$\Omega_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1,\ 0 < x_2 < a(x_1) + g(x_1/\epsilon)\}.$$

where $a_\delta(x_1)$ is a piecewise constant function satisfying

$$\|a_\delta - a\|_{L^\infty(0,1)} \leq \delta.$$



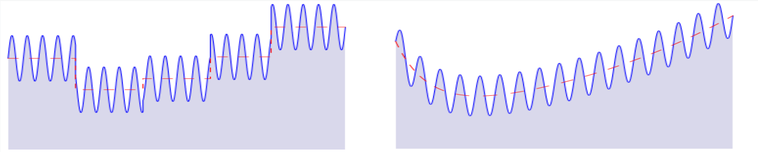
$$\downarrow (\epsilon \longrightarrow 0)$$

$$\downarrow$$

$$(\text{Equation})_\delta \quad \overset{\delta\to 0}{\longrightarrow} \quad (\text{Equation})_0$$

$$(\text{Eq})_\delta : \quad \int_0^1 \left(q_\delta |Y_\delta^*| u_0' \varphi' + |Y_\delta^*| u_0 \varphi \right) dx_1 = \int_0^1 |Y_\delta^*| f \varphi dx_1$$

and $\quad q_\delta = q_\delta(x), \quad Y_\delta^* = Y_\delta^*(x)$



$$\downarrow (\epsilon \to 0) \qquad\qquad \downarrow ?$$

$$\downarrow \qquad\qquad \downarrow ?$$

$$(\text{Equation})_\delta \quad \xrightarrow{\delta \to 0} \quad (\text{Equation})_0$$

$$(\text{Eq})_\delta : \quad \int_0^1 \left(q_\delta |Y_\delta^*| u_0' \varphi' + |Y_\delta^*| u_0 \varphi \right) dx_1 = \int_0^1 |Y_\delta^*| f \varphi dx_1$$

$$\text{and} \qquad q_\delta = q_\delta(x), \quad Y_\delta^* = Y_\delta^*(x)$$

The convergence $(Eq)_\delta \stackrel{\delta\to 0}{\longrightarrow} (Eq)$ follows from:

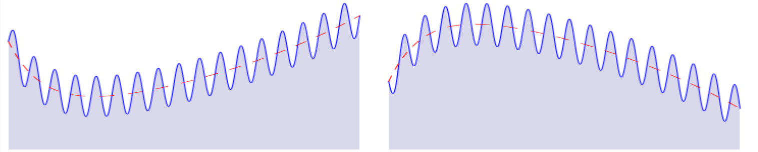
$$Y_\delta^*(x) = \{(y,z) \in \mathbb{R}^2 \ : \ 0 < y < L, 0 < z < a_\delta(x) + g(y)\},$$

$$q_\delta(x) = \frac{1}{|Y_\delta^*(x)|} \int_{Y_\delta^*(x)} \left(1 - \frac{\partial X_\delta}{\partial y_1}(x, y_1, y_2)\right) dy_1 dy_2$$

and $X_\delta(x,\cdot,\cdot)$ satisfies

$$\begin{cases} -\Delta_{y,z} X_\delta(x,y,z) = 0 \quad \text{in } Y_\delta^*(x) \\ BC_\delta(X_\delta) = 0 \end{cases}$$

We are going to show a continuous dependence result on the function $a(x)$ for the solutions in the thin domains:



Ω_ϵ $\hat{\Omega}_\epsilon$

$$\Omega_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1,\ 0 < x_2 < a(x_1) + g(x_1/\epsilon)\}.$$
$$\hat{\Omega}_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < 1,\ 0 < x_2 < \hat{a}(x_1) + g(x_1/\epsilon)\}.$$

with $\alpha_0 \leq a(x), \hat{a}(x) \leq \beta_0$.

Denote by u_ϵ and $\hat{u}_\epsilon$ the solutions of

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f_\epsilon & \text{in } \Omega^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial x_1} N_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2} N_2^\epsilon = 0 & \text{on } \partial\Omega^\epsilon \end{cases}$$

$$\begin{cases} -\dfrac{\partial^2 \hat{u}^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 \hat{u}^\epsilon}{\partial {x_2}^2} + \hat{u}^\epsilon = f_\epsilon & \text{in } \hat{\Omega}^\epsilon \\ \dfrac{\partial \hat{u}^\epsilon}{\partial x_1} \hat{N}_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial \hat{u}^\epsilon}{\partial x_2} \hat{N}_2^\epsilon = 0 & \text{on } \partial\hat{\Omega}^\epsilon \end{cases}$$

with $f_\epsilon \in L^2(\mathbb{R}^2)$.

Theorem

There exists a function $\rho(\delta) \to 0$ as $\delta \to 0$ such that

$$\|u_\epsilon - \hat{u}_\epsilon\|^2_{H^1_\epsilon(\Omega_\epsilon \cap \hat{\Omega}_\epsilon)} + \|u_\epsilon\|^2_{H^1_\epsilon(\Omega_\epsilon \setminus \hat{\Omega}_\epsilon)} + \|\hat{u}_\epsilon\|^2_{H^1_\epsilon(\hat{\Omega}_\epsilon \setminus \Omega_\epsilon)} \leq \rho(\delta)$$

uniformly for all

- $\epsilon \in (0, \epsilon_0)$
- pieciwise C^1 functions a, $\hat{a}$ with $\|a - \hat{a}\|_{L^\infty(0,1)} \leq \delta$, $\alpha_0 \leq a(x), \hat{a}(x) \leq \alpha_1$
- $f_\epsilon \in L^2(\mathbb{R}^2)$, $\|f_\epsilon\|_{L^2(\mathbb{R}^2)} \leq 1$

Remark: The space, $H^1_\epsilon(U) = H^1(U)$ with the norm

$$\|u\|^2_{H^1_\epsilon(U)} = \|u\|^2_{L^2(U)} + \|u_{x_1}\|^2_{L^2(U)} + \frac{1}{\epsilon^2}\|u_{x_2}\|^2_{L^2(U)}$$