Routing for Energy Minimization in the Speed Scaling Model

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Model and problem

Network optimization problem

Problem statement

- Undirected network $G = (V, E)$, with $n$ vertices and $m$ edges.
- Set of traffic demands $(s_1, t_1), \ldots, (s_d, t_d)$.
- A cost function $f_e(x)$ for each $e \in E$.
- Objective: Route each demand in $G$ so that the total cost $\sum_{e} f_e(x_e)$ is minimized.

Applications:

- Installation of oil pipes.
- Communication networks bandwidth provisioning.

Usually show economy of scale.
Network optimization problem

Formulation with linear constraints

\[
(P_0) \quad \min \sum_{e} f_e(x_e)
\]

subject to

\[
x_e = \sum_{i} y_{i,e} \quad \forall e
\]

\[
y_{i,e} \in \{0, 1\} \quad \forall i, e
\]

\[
y_{i,e} : \text{ flow conservation}
\]

\[y_{i,e}: \text{ Indicator of whether demand } i \text{ is routed through edge } e.\]

\[x_e: \text{ Load in edge } e.\]
Cost functions

**Linear functions**

- Cost functions $f_e(x) = \mu \cdot x$.
- Shortest path routing is optimal.
Cost functions

Subadditive functions

- Cost functions \( f_e(x) + f_e(x') > f_e(x + x') \).
- Well-studied optimization problem: Buy-at-bulk.
- \( f_e(x) \) uniform: \( O(\log n) \)-approx ratio, \( \Omega(\log^{1/4} n) \) hardness.
- \( f_e(x) \) nonuniform: \( O(\log^3 n) \)-approx ratio, \( \Omega(\log^{1/2} n) \) hardness.
Other functions

- Functions that are not linear nor subadditive have been less studied.
- Energy consumption curves as function of processing speed fall in this class.
Why do we care?

Reasons for energy savings in ICT

- Growing energy bill (1.5% of US consumption in 2006).
- Pressure from governments and society to reduce carbon emissions.

Potential savings: 80% reduction at the access layer. 40% reduction in the network core.

How to save

- **Speed scaling**: Adapt the speed (hence energy) of network elements to the carried load.
- **Globally** optimize energy consumption in the network.
### Contributions

**Superadditive functions**
- NP-hard with **no finite approximation**.

**Polynomial functions**
- \( f_e(x) = \mu e x^\alpha \): Constant-approximation.

**Polynomial functions with startup**
- \( f_e(x) = \begin{cases} 
0 & \text{if } x = 0, \\
\sigma_e + \mu e x^\alpha & \text{if } x > 0.
\end{cases} \)
- \( \Omega(\log^{1/4} n) \)-hardness, for \( \alpha > 1 \).
- \( O((\max_e \{\sigma_e/\mu_e\})^{1/\alpha} + 1) \)-approximation.
- \( O(d) \)-approximation, for \( d \) demands.
Inapproximability of superadditive functions

- Reduction from edge-disjoint path (EDP):
  - Given network $G$ and demands $(s_i, t_i)$, decide if there are edge-disjoint routes.
  - NP-hard

- Consider monotone function $f_e(1) = 0$, $f_e(2) > 0$.
- If disjoint routes, $\sum_e f_e(x_e) = 0$, otherwise $\sum_e f_e(x_e) > 0$.
- Any bounded approximation returns 0 iff disjoint routes.
Approximation for polynomial functions

**Polynomial functions**

Functions of the form $f_e(x) = \mu_e x^\alpha$.

**Approach**

- Relax the integrality constrain $y_{i,e} \in \{0, 1\}$.
- Solve the problem.
- Apply randomized rounding.
Approximation for polynomial functions

Relaxed formulation

Convex program

\[
(P_1) \quad \min \sum_e \mu_e x_e^\alpha
\]

subject to

\[
x_e = \sum_i y_{i,e} \quad \forall e
\]

\[
y_{i,e} \in [0, 1] \quad \forall i, e
\]

\[
y_{i,e} : \text{flow conservation}
\]

Polynomial-time solvable!
Approximation for polynomial functions

Approximation algorithm

Raghavan-Thompson randomized rounding

- Solve the convex program \((P_1)\), and obtain optimal fractional solution \((x^*, y^*)\).
- For each demand \(i\), iteratively decompose the unit flow of \(i\) into paths according to the \(y_{i,e}^*\) values:
  - Assign to path \(p\) weight \(w_p = \min_{e \in p} y_{i,e}^*\).
  - For each \(e \in p\), decrement \(y_{i,e}^*\) by \(w_p\).
- For each demand \(i\), several flow paths \(p_{i,1}, p_{i,2}, p_{i,3}, \ldots\) are obtained such that \(\sum_j w_{p_{i,j}} = 1\).
- For each demand \(i\), randomly choose one of the paths using the weights as probabilities. Let the output be \((\hat{x}, \hat{y})\).
Approximation for polynomial functions

**Integrality gap of direct approach**

- \( f_e(x) = x^\alpha \) and only one demand \((s, t)\).
- Optimal fraction solution is \( x_1 = x_2 = \cdots = x_m = 1/m \).
- Optimal integral solution is \( x_1 = 1, x_2 = \cdots = x_m = 0 \).

The integrality gap is

\[
\frac{1}{m \cdot f_e(1/m)} = \frac{1}{m \cdot (1/m)^\alpha} = m^{\alpha - 1}
\]
The previous gap comes from the behavior of $f_e(x)$ in $[0, 1]$. We can use function $\mu_e x$, in this interval, since they agree in $0$ and $1$. We define a new cost function: $g_e(x) = \mu_e \max\{x^\alpha, x_e\}$

![Graph showing the comparison of $f_e(x)$, $\mu_e x$, and $g_e(x)$]
Approximation for polynomial functions

New formulation

Convex program

\[(P'_1) \quad \min \sum_{e} g_e(x_e)\]

subject to

\[x_e = \sum_{i} y_{i,e} \quad \forall e\]

\[y_{i,e} \in [0, 1] \quad \forall i, e\]

\[y_{i,e} : \text{flow conservation}\]

\[\sum_{e} g_e(x_e) \text{ and } \sum_{e} f_e(x_e) \text{ have the same integral optimal.}\]
Approximation for polynomial functions

Constant approximation

**Lemma**

Randomized rounding the optimal fractional solution $x_e^*$ guarantees that $E[f_e(\hat{x}_e)] \leq \gamma g_e(x_e^*)$ for all $e$.

Consider two cases:

- If $x_e^* \leq 1$, partition the possible values of $\hat{x}_e$ into the intervals $[0, 1), [1, 2), \ldots [2^j, 2^{j+1}), \ldots$, and apply a Chernoff bound to $\Pr[\hat{x}_e \geq 2^j]$.

- If $x_e^* > 1$, partition the possible values of $\hat{x}_e$ into the intervals $[0, x_e^*), [x_e^*, 2x_e^*), \ldots [2^j x_e^*, 2^{j+1} x_e^*), \ldots$, and apply a Chernoff bound to $\Pr[\hat{x}_e \geq 2^j x_e^*]$. 
Approximation for polynomial functions

Constant approximation

**Theorem**

Randomized rounding provides a $\gamma$-approximation in the expected value of the total energy cost. For any constant $c$, randomized rounding provides a $c\gamma$-approximation with probability at least $1 - 1/c$.

This result also applies when demands are uniform and can be generalized to the case when demands are non-uniform.

**Theorem**

For nonuniform demands, randomized rounding can be used to achieve a $O(\log^{\alpha-1} D)$-approximation, where $D = \max_i d_i$. 
Polynomial functions with startup: Hardness

\[ f_e(x) = \begin{cases} 
0 & \text{if } x = 0, \\
\sigma_e + \mu_e x^\alpha & \text{if } x > 0. 
\end{cases} \]

**Theorem**

For \( \alpha > 1 \) there is a uniform \( f_e(x) \) such that no algorithm has \( \log^{1/4} n \) approximation ratio unless \( NP \subseteq ZPTIME(n^{\text{polylog} n}) \).

The proof constructs an instance of the routing problem that encodes a 3CNF(5) formula. A \( \log^{1/4} n \) approximation algorithm could be used to probabilistically decide satisfiability of a subset of clauses, contradicting the PCP theorem.
Let us use the same functions $g_e(x)$ as before and consider the following convex program.

Convex program

\begin{align*}
(P_2) & \quad \min \sum_e \sigma_e z_e + g_e(x_e) \\
\text{subject to} & \\
x_e &= \sum_i y_{i,e} \quad \forall e \\
z_e &= \max_i \{y_{i,e}\} \quad \forall e \\
y_{i,e} &\in [0, 1] \quad \forall i, e \\
y_{i,e} &: \text{flow conservation}
\end{align*}
Randomized rounding provides a $O(d)$-approximation in the expected value of the total energy cost.

Follows from

- $E[f_e(\hat{x}_e)] \leq \gamma g_e(x_e^*)$
- $E[\hat{z}_e] = Pr(\hat{z}_e = 1) = 1 - Pr(\hat{y}_{i,e} = 0 \text{ for all } i)$
  $= 1 - \prod_i (1 - y_{i,e}^*) \leq \sum_i y_{i,e}^* \leq dz_e^*$.

For non-uniform demands, $O(d + \log^{\alpha-1} D)$ approximation ratio can be achieved, where $D = \max_i d_i$. 

$O(d)$-approximation
Approximation with fitting function

Let \( s_e = (\sigma_e / (\alpha - 1) \mu_e)^{1/\alpha} \). Define

\[
\beta_e = \begin{dcases}
\sigma_e + \mu_e & \text{if } s_e < 1, \\
\alpha \mu_e (\sigma_e / (\alpha - 1) \mu_e)^{1 - 1/\alpha} & \text{if } s_e \geq 1.
\end{dcases}
\]

\[
ge_e(x) = \begin{dcases}
\beta_e x & \text{if } x \in [0, \max(1, s_e)), \\
\sigma_e + \mu_e x^{\alpha} & \text{if } x \geq \max(1, s_e).
\end{dcases}
\]
Formulation

Convex programming

\((P'_1)\) \quad \min \sum_e g_e(x_e)

subject to

\[ x_e = \sum_i y_{i,e} \quad \forall e \]

\[ y_{i,e} \in [0, 1] \quad \forall i, e \]

\[ y_{i,e} : \text{flow conservation} \]
Randomized rounding to the fractional solution obtained from the convex program ($P'_1$), guarantees
$$E[f_e(\hat{x}_e)] \leq O(1 + (\sigma_e/\mu_e)^{1/\alpha}) \cdot g_e(x_e^*),$$ for each link $e$. Hence, $O((\max_e{\sigma_e/\mu_e})^{1/\alpha} + 1)$-approximation.

The proof considers three cases
- $x_e^* \leq 1$. Implies $E[f_e(\hat{x}_e)] \leq O(1 + (\sigma_e/\mu_e)^{1/\alpha}) \cdot g_e(x_e^*)$.
- $x_e^* \geq \max(1, s_e)$. Implies $E[f_e(\hat{x}_e)] \leq \gamma \cdot g_e(x_e^*)$.
- $x_e^* \in (1, s_e)$. Implies $E[f_e(\hat{x}_e)] \leq O(1 + (\sigma_e/\mu_e)^{1/\alpha}) \cdot g_e(x_e^*)$.

For non-uniform demands a similar result with an additional $O(\log^{\alpha-1} D)$ factor, where $D = \max_i d_i$, can be shown.
We have explored a min-cost integer routing problem for cost functions that model energy consumption.
Positive and negative results have been shown for polynomial functions with and without startup cost.
While pure polynomial functions can be efficiently routed (constant approximation), reaching the polylogarithmic lower bound seems to be hard when there are startup costs.
Simulation results (omitted) show that the proposed algorithms work well in practice.
Future work

- For polynomial functions with startup cost, we would like to get approximation ratios independent of the functions and the demands, and as close as possible to the polylogarithmic bound.
- Other classes of functions can be explored.
- Additional restrictions in the solutions can be imposed (discrete speeds, capacity limits).
- Other models of energy savings can be explored: E.g., powering down the links.
Thanks!

Thank you!!