Convex integration for the Monge-Ampére system and other geometric problems.

The Monge-Ampère system (MA) is the multi-dimensional version of the Monge-Ampère equation arising from the prescribed curvature problem, closely related to the problem of isometric immersions and the dimension reduction of thin shells. Namely, given $F : \omega \to \mathbb{R}^{d^4}$ on $\omega \subset \mathbb{R}^d$, we look for v in:

$$v : \omega \to \mathbb{R}^k, \qquad \mathfrak{Det}\,\nabla^2 v \doteq \big[\langle \partial_i\partial_s v, \partial_j\partial_t v\rangle - \langle \partial_i\partial_t v, \partial_j\partial_s v\rangle\big]_{i,j,s,t:1\ldots d} = F \quad \text{in } \omega. \tag{MA}$$

Indeed, the Riemann curvatures of the metrics generated by the immersions $\{u^\epsilon = (id_d, \epsilon v)\}_{\epsilon\to 0}$, are:

$$Riem\big((\nabla u)^T\nabla u\big) = Riem\big(\mathrm{Id}_d + \epsilon^2(\nabla v)^T\nabla v\big) = -\frac{\epsilon^2}{2}\mathfrak{C}^2\big((\nabla v)^T\nabla v\big) + o(\epsilon^2) = \epsilon^2\mathfrak{Det}\,\nabla^2 v + o(\epsilon^2).$$

When $d = 2$, $k = 1$, the system (MA) reduces to the classical Monge-Ampére equation $\det \nabla^2 v = f$ as the prescription of the Gaussian curvature of the shallow surface given by the graph of ϵv, whereas (MA) prescribes the full Riemann curvature of the shallow manifold $\{(x, \epsilon v(x));\ x \in \omega\} \subset \mathbb{R}^{d+k}$.

A necessary condition for (MA) to be well posed is that $F \in Range(\mathfrak{C}^2)$, or $F = -\mathfrak{C}^2(A)$ for some given $A : \omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$. Observing that $Kernel(\mathfrak{C}^2)$ consists of symmetrized gradients, (MA) takes its weak formulation, called the Von Kármán system, in which we look for v, w such that:

$$v : \omega \to \mathbb{R}^k, \quad w : \omega \to \mathbb{R}^d, \qquad \frac{1}{2}(\nabla v)^T\nabla v + \mathrm{sym}\,\nabla w = A \quad \text{in } \omega. \tag{VK}$$

When $d = 2$, $k = 1$, the left hand side of (VK) is known in the theory of elasticity as the the Von Kármán stretching content whose energy measures the stretching of a thin film with midplate ω, subject to the out of plane displacement v and the in plane displacement w.

Closely related to (MA) and (VK) is the problem of finding an isometric immersion u of the given Riemannian metric $g : \omega \to \mathbb{R}^{d\times d}_{\mathrm{sym},>}$, into a higher dimensional space $\mathbb{R}^{d+k}$:

$$u : \omega \to \mathbb{R}^{d+k}, \qquad (\nabla u)^T\nabla u = g \quad \text{in } \omega. \tag{II}$$

Indeed, (II) yields (VK) when equating the leading order terms in the family of metrics $\{\mathrm{Id}_d + 2\epsilon^2 A\}_{\epsilon\to 0}$ and the metrics generated by immersions $\{\bar u^\epsilon = (id_d + \epsilon^2 w, \epsilon v)\}_{\epsilon\to 0}$. Thus the three problems (MA), (VK) and (II) are intrinsically related.

This course will concern the ongoing study of existence, regularity, and multiplicity of solutions to systems (MA), (VK), (II) through the method of convex integration, building on the prior fundamental results due to Nash, Kuiper, Borisov and more recently Conti, Delellis and Szekelyhidi. We will also explore relation to the scaling of the non-Euclidean energies of elastic deformations and the quantitative isometric immersion problem. The course will be self-contained, with outline as follows:

Lecture 1: The geometric systems (II), (MA), (VK). Classical results on existence and regularity of their solutions. Non-uniqueness of solutions in the absence of notion of curvature.

Lecture 2: The Nash-Kuiper scheme and Hölder regularity. The defect decomposition lemmas. Convex integration technique via Kuiper's corrugations and Nash's spirals.

Lecture 3: Källen's iteration. Step, stage and convergence strategies. Iteration on codimension. Flexibility of the Poznyak's result. The special case of $d = 2$. Conjectures for $d > 2$.

Lecture 4: Rigidity results. Relation to Euler's equations, the role of the commutator estimate and the degree lemma.

Lecture 5: Relation to nonlinear elasticity. Scaling of the non-Euclidean elastic energies of prestressed thin films. Large and weak prestress regimes. The quantitative isometric immersion problem.